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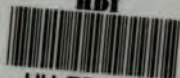
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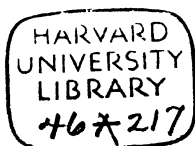
BY
V. KARAPETOFF

PART V.
PROBLEMS ON
ELECTRICAL ENGINEERING

FIRST EDITION
FIRST THOUSAND

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PREFACE TO PART V.

THIS last part of the work contains problems in electrical engineering, and these, as well as the problems of the other parts, have been selected to illustrate the important principles of analytics and calculus. The several parts are independent of one another, and contain problems selected from the following engineering subjects:

Part I. Machine Design.

Part II. Hydraulics.

Part III. Thermodynamics.

Part IV. Mechanics of Materials.

Part V. Electrical Engineering.

A student or an engineer who wishes to review calculus or analytic geometry, or who wishes to acquire facility in applications of higher mathematics to engineering problems, may begin with any one of the five parts. He will probably first read the part which deals with his own branch of engineering, and then take the others with which he is less familiar. In this way every reader can arrange the parts according to an ascending degree of difficulty.

The author's views on teaching mathematics to engineering students may be found in the Preface to Part I and in the Dialogue following that preface. Reference may also be made to Part I for a list of works on mathematics and for an Appendix entitled, "What a Senior in Engineering ought to know about Mathematics."

With this fifth part is completed a work which represents the result of many years of study and effort on the part of the author to inspire his fellow engineers

with love for mathematics and to impress upon them the simplicity, the importance, and the efficacy of the mathematical method. In 1898, John Perry's "Calculus for Engineers," came to the author's notice and since then he has agreed with the writer that in teaching analytics and calculus to engineering students the instructor should limit himself at first to a comparatively small number of curves, functions, and transformations, but that each of these must be explained and *made concrete* in a number of engineering problems carried through to their practical conclusions.

A natural mathematician enjoys abstract relationships from which the concrete applications are obtained as special cases. But most engineers are not natural mathematicians, and they learn mathematics much better by beginning with the more specific cases and gradually generalizing them as the number of these cases increases. Such is their type of mind, a type that is valuable in practical work, and it is unpsychological to train them in the opposite way. It would seem that either our instructors in mathematics must come to understand this difference between their temperament and that of their students, or else mathematics in engineering colleges will have to be taught by engineers.

Another important and much needed improvement in teaching mathematics to our engineering students would be to carry the subject through all the four or five undergraduate years, and not discontinue it at the time when the student is just beginning to understand what it is all about. I believe it was the eminent electrician Dr. C. P. Steinmetz, who expressed the correct psychological idea that a scientific method which is worth studying at all is worth continuing throughout the college course; he meant by this that

the threads of thought once grasped should not be allowed to break.

Combining therefore the two foregoing requirements, namely, from the specific to the general, and the continuity of training, we get the following outline of a rational course in mathematics for engineering students:

First Year. Applications of elementary mathematics to engineering problems; additional chapters on solid geometry and trigonometry, with applications.

Second Year. The straight line and a few simple curves in analytics; differentiation and integration of a few simple functions; numerous applications to engineering problems.

Third Year. Elementary analytics and calculus, with applications to engineering and physics.

Fourth Year. Advanced chapters from analytics and calculus; generalization of the information previously gained, with more stringent proofs; a few simple differential equations.

Fifth Year (Optional). Differential equations, hyperbolic functions, theory of probabilities, or any other branch of higher mathematics that the student is interested in or that may be of use to him in his specialty.

Such an arrangement of the subject matter may be called concentric, because it may be symbolically represented by a series of concentric circles, with their zones covering more and more ground, and each extending in all directions.*

Of course, a continuous schedule in mathematics would mean some rearrangement in the order of the other studies, but such a rearrangement is highly desirable anyway, if one accepts the principle of leading

* See the author's paper "On the Concentric Method of Education in Engineering," Proc. Soc. Promotion Eng'g Education, Vol. 16 (1908), p. 258.

from the specific and the concrete, to the general and the abstract. The descriptive courses in engineering, as well as shop, drafting, and simple laboratory experiments, would have to be more concentrated in the first two years of the course, and the more theoretical courses taught later on the basis of the tangible engineering information already acquired. In particular, physics and mechanics with their general laws ought to be shifted towards the end of the course when the student is mature enough to appreciate them. A few simple experimental laws of nature which are used in engineering can be easily given in the corresponding technical courses. As a matter of fact, teachers of engineering actually find it necessary to repeat these laws to their students in spite of the preceding courses in physics and mechanics.

Originally Part V was planned to be much larger, but in the meanwhile the author has published two other books of a mathematical character, "The Electric Circuit" and "The Magnetic Circuit" (McGraw-Hill Book Co.), and rather than to duplicate the text he limited himself in this book to electrical problems of a somewhat different type. The principal difficulty in selecting problems from electrical engineering for a book of this character consists in avoiding relationships the physical significance of which may not be clear to the student. Since we have no electric or magnetic sense among the five external senses, these relations are bound to remain more intangible than the phenomena dealt with in mechanics, hydraulics, thermodynamics, or chemistry.

Mr. A. C. Stevens, M.E., instructor in Cornell University, has read the proofs of this volume and to him the author is under obligation for the care with which the printing was done.

CORNELL UNIVERSITY, ITHACA, N. Y.
May, 1916.

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PART V.
PROBLEMS ON
ELECTRICAL ENGINEERING.

ENGINEERING MATHEMATICS.

CHAPTER I.

RESISTANCE OF NON-CYLINDRICAL CONDUCTORS AND MAGNETIC PATHS.

THE three fundamental quantities with which an electrical engineer has constantly to deal are: electromotive force, current, and electrical resistance. The intrinsic nature of electrical phenomena being unknown, it is hardly possible to convey to a beginner a clear idea of these quantities, though to an electrical engineer or a physicist these quantities become real through constant usage. Assuming the flow of electricity to be analogous to the flow of a liquid through a pipe, an electric current may be compared to the quantity of the liquid passing per second through a cross section of the pipe. The resistance of the conductor may be likened to the friction between the fluid and the pipe. The electromotive force, or the difference of potential between two cross sections of a conductor, is then analogous to the difference in pressure between the corresponding cross sections of the pipe; this loss of pressure being necessary for overcoming the friction along the pipe.

Or, comparing the flow of electricity to that of heat along a rod, the electromotive force corresponds to the difference of temperatures, being the cause of the flow; the current is analogous to the quantity of heat

flowing through a cross section of the rod per unit time; the electric resistance corresponds then to the resistance of the rod to the flow of heat; in other words, it is the reciprocal of the thermal conductance of the rod.

Ohm found experimentally that for a given metallic conductor, at a constant temperature, the current is proportional to the applied electromotive force. For instance, let an electromotive force of 10 volts between the ends of a conductor produce a current of 4 amperes in the conductor. Then an electromotive force of 20 volts will cause a current of 8 amperes to flow. This relation is known as Ohm's law, and is generally written in the form

$$E = RI, \quad (1)$$

where E is the difference of potential, or the voltage between the ends of the conductor, I is the current and R is the resistance of the conductor, or the coefficient of proportionality between E and I .

With long cylindrical rods, strips, wires, and in general where electricity flows in parallel stream lines uniformly distributed throughout the cross section of a conductor, the resistance is proportional to the length l of the conductor and is inversely proportional to its cross section q . Thus,

$$R = \rho \cdot \frac{l}{q}, \quad (2)$$

and Eq. (1) becomes

$$E = \rho \cdot \frac{l}{q} I. \quad (3)$$

In these equations ρ is a coefficient which depends upon the material of the conductor and upon its tem-

perature. The coefficient ρ is called the *resistivity* of the material.

For copper and aluminum wires, and for some other metals commonly used as electrical conductors, tables are available in various handbooks from which resistance may be readily obtained for standard cross sections and lengths. Thus, for instance, the resistance of a copper wire having a cross section of 1 sq. mm., and 1 meter long, at a standard temperature of 25° C., is about 0.018 ohm. This is also the value of ρ for copper at this temperature, provided that lengths are measured in meters and cross sections in square millimeters, because then in Eq. (2) $l = 1$ and $q = 1$. Therefore, for a trolley wire 3 km. long and of 70 sq. mm. in cross section, the resistance is, according to Eq. (2),

$$R = 0.018 \times 3000/70 = 0.77 \text{ ohm.}$$

When a current of, say, 40 amperes is flowing through this wire the voltage across its ends is $40 \times 0.77 = 30.8$ volts.

Ohm's law may be expressed in words by saying that in a metal conductor the electrical *effect* (current I) is proportional to the *cause* (voltage E). There are some other natural phenomena in which the effect is proportional to the cause; for instance,

(1) In a sand filter, under certain conditions, the quantity of the liquid which passes through the sand is proportional to the applied hydraulic pressure or head.

(2) In a homogeneous wall, the quantity of heat which passes across it in unit time is proportional to the difference in temperature of the two surfaces.

(3) In the absence of iron, the magnetic flux produced by a current in a given coil is proportional to that current.

(4) In dielectrics (commonly known as insulating substances) a displacement of electricity, or dielectric flux, is proportional to the voltage which produces it.

For all these cases equations may be written which are mathematically identical with Eqs. (1), (2) and (3), although their physical significance is entirely different. Some authors even speak of Ohm's law in application to the flow of heat, or to magnetic phenomena, etc. Therefore, the problems solved in this chapter apply to several other classes of phenomena than that of the flow of current through metallic conductors, and the student will do well to make clear to himself the physical significance and the practical importance of the results in application to these different phenomena.

In the solution of the problems that follow, the student will need the following theorem: When two or more resistances are connected in series the equivalent resistance is equal to the sum of the component resistances; or

$$R_{eq} = \Sigma R. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

A proof of this proposition will be found in almost any elementary text-book on electricity.*

In some problems resistances are connected in parallel; in this case it is convenient to consider the reciprocal of each resistance, called the *conductance*. We have, from Eq. (2), the following expression for the conductance g of a conductor:

$$g = \frac{1}{R} = \gamma \cdot \frac{q}{l}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where γ is equal to $1/\rho$ and is called the conductivity of the material. By analogy with Eqs. (1) and (3) we have:

$$I = Eg, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

* See, for instance, the author's "Electric Circuit," p. 7.

and

$$I = \gamma \frac{q}{l} E. \quad (7)$$

The following theorem applies to conductances in parallel: When two or more conductors are connected in parallel, the equivalent conductance is equal to the sum of the component conductances;* or

$$g_{eq} = \Sigma g. \quad (8)$$

As a general rule, the student will find it convenient to use resistances when conductors are connected in series, and to use conductances when conductors are connected in parallel.

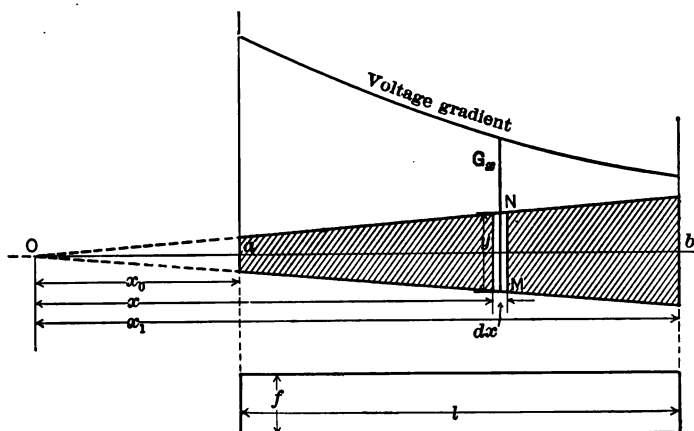


FIG. 1. — A wedge-shaped conductor.

Prob. 1. — Find the resistance of a wedge-shaped conductor (Fig. 1) of length l and width f , the height at one end being a , and at the other b . *Solution.* — Let the traces of the two converging planes meet at O , at a distance x_0 from a . Consider the conductor to consist of infinitesimal prismatic layers, such as MN . If y is the height of such a layer, its resistance is $\rho dx/(fy)$. According to

* For a proof of this proposition see, for instance, the author's "Electric Circuit," p. 8.

Eq. (4), the total resistance of the conductor is equal to the sum of the resistances of the infinitesimal slices, or

$$R = \int_{x_0}^{x_1} \rho \, dx / (fy) (9)$$

In order to be able to integrate this equation, y must be expressed as a function of x . We have from the figure that $x_0 : a = x : y$, so that $y = ax/x_0$. Substituting this value in Eq. (9) and integrating we find that

$$R = \frac{\rho x_0}{fa} \text{Ln} \frac{x_1}{x_0}, \quad (10)$$

where the symbol Ln stands for the natural logarithm. The constants x_0 and x_1 may be eliminated by using the conditions $x_1/x_0 = b/a$, and $x_0/l = a/(b-a)$.

Prob. 2. — Find the resistance of a conductor shaped like a truncated cone, the length of the axis being l , and the radii of the extreme cross sections r_0 and r_1 . *Hint.* $dR = \rho \, dx / (\pi r^2)$, where the variable r must be expressed through x from the similar triangles, as in Prob. 1.

Prob. 3. — On the basis of the answer to Prob. 2, show that the resistance of a tapering circular conductor is equal to that of a cylindrical conductor of the same material and length, the cross section of the cylinder being the geometric mean of the extreme sections of the cone.

Prob. 4. — A conductor has the shape of a hollow truncated cone, the thickness of the wall being the same at all places. Deduce an expression for its resistance.

Prob. 5. — A conductor has the shape of a hollow truncated cone; the radii at one end are r_0 and r_1 , and at the other end r_2 and r_3 . Find an expression for its resistance.

Prob. 6. — A conductor is formed by the revolution of an ordinary parabola (Fig. 2) about axis OX . The extreme cross sections are q_0 and q_1 , the length is l . What is the resistance of the conductor? *Hint.* $y_0^2 = q_0/\pi = 2 \, px_0$; $y_1^2 = q_1/\pi = 2 \, px_1$; $y^2 = q/\pi = 2 \, px$; $x_1 - x_0 = l$.

Prob. 7. — A conductor has the shape of a body of revolution formed by the rotation of a given curve $y = f(x)$ about the X -axis. Write down an expression for the electrical resistance of the conductor between two given cross sections.

Prob. 8. — Prove that the curve of voltage gradient G_x in Prob. 1 is a hyperbola. The voltage gradient is defined as the voltage drop per unit length of the conductor. *Solution.* — The resistance of an infinitesimal layer being $\rho \, dx/(fy)$, the voltage drop across this

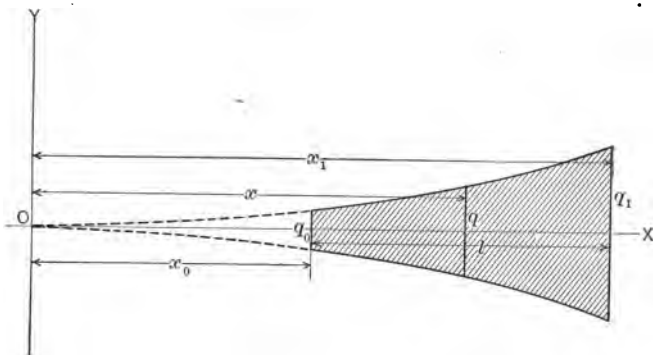


FIG. 2. — A conductor formed by revolution of a parabola.

layer is $de = i\rho \, dx/(fy)$, where i is the current flowing through the conductor. By the definition of the voltage gradient, $G_x = de/dx = i\rho/(fy)$. Substituting the value of y found in the solution of Prob. 1, we find $G_x = i\rho x_0/(fax)$, which is a hyperbola of the form $G_x \cdot x = \text{const.}$

Prob. 9. — Find the equations of the curves of voltage gradient in Prob. 2 and in Probs. 4 to 7.

Prob. 10. — Find the resistance of the insulation of a single-core concentric cable (Fig. 3) [of length l]. *Solution.* — Let the core be connected to the positive pole and the sheath or lead cover to the negative pole. The insulation being imperfect, there is a tendency for the

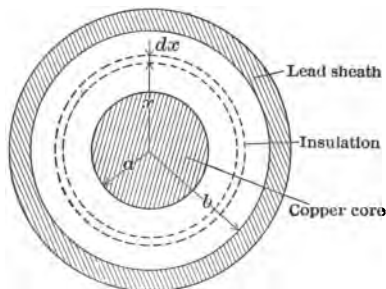


FIG. 3. — A single-core concentric cable.

current to flow from the core into the sheath. The resistance of the insulation may be resolved into the resistances of infinitesimal concentric cylindrical layers. Consider such a layer of radius x and of thickness dx . According to formula (2), the resistance

of this shell is $dR = \rho \, dx / (2 \pi x l)$. Integrating this expression between the limits a and b , we find $R = (\rho / 2 \pi l) \text{Ln} (b/a)$.*

Prob. 11. — The insulation of the cable shown in Fig. 3 consists of two concentric layers of resistivities ρ_1 and ρ_2 respectively. The separating surface between the two is of radius c . Write down an expression for the resistance of the whole insulation.

Prob. 12. — Referring to Prob. 10, if the total voltage between the core and the sheath is E , what is the voltage e between two points in the insulation at distances r_1 and r_2 from the center? *Hint.* $de = i \rho \, dx / (2 \pi x l)$, where i is the unknown leakage current. Integrate this expression between the limits r_1 and r_2 , and eliminate the unknown current using the condition that $e = E$, when $r_1 = a$ and $r_2 = b$.

Prob. 13. — The diameter of the core of a concentric cable is 9 mm., the thickness of the insulation is 4 mm.; the leakage current per km. length is i . Can the leakage current be reduced to $\frac{1}{2} i$ by increasing the thickness of the insulation to 8 mm.?

Prob. 14. — Let Fig. 3 represent concentric spheres, instead of cylinders. Solve Probs. 10 to 13 for this case.

Prob. 15. — The final answer to Prob. 1 is of the form

$$R = \text{Const.} \times \frac{\text{Ln} (b/a)}{b - a}. \quad \dots \quad (11)$$

For a conductor of constant thickness, $b = a$, and the foregoing expression assumes an indeterminate form $0/0$. Show that the true value of the ratio is 1. *Hint.* — Let $b = (1 + \theta) a$, where θ is a small quantity, and evaluate the indeterminate form according to the rule given in the differential calculus.

Prob. 16. — When b in formula (11) differs but little from a , numerical computations become unsatisfactory, because a small error in either value may affect the result to a considerable degree. Show that it is better in this case to expand R into an infinite series. *Hint.* — Expand $\text{Ln} (1 + \theta)$ in powers of θ , and divide each term by θ .

Prob. 17. — An electric current flows between electrodes A and B (Fig. 4) through a medium of comparatively high resistivity. Electrode A consists of "teeth," electrode B is a large mass of

* A similar method may be used in deriving the electrostatic capacity of a concentric cable; see the author's "Electric Circuit," Chap. XVI.

metal limited by a plane. Both electrodes are assumed to extend indefinitely in the direction perpendicular to the plane of the paper, so that the problem is a two-dimensional one. Determine the resistance of the medium between the electrodes, within the limits

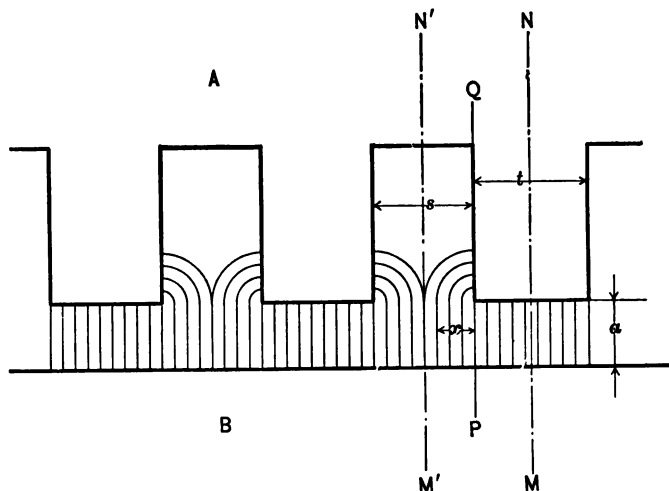


FIG. 4. — Stream lines between teeth and a solid pole.

MN and $M'N'$, for a length l in the direction perpendicular to the plane of the paper. The stream lines, or the lines of flow of current, are assumed to consist of straight lines and concentric quadrants.*

Solution. — Since the individual stream lines are electrically connected in parallel, it is more convenient to consider their conductances, according to Eqs. (5) and (8). In the region between MN and PQ the conductance is $g_t = \gamma \cdot \frac{1}{2} tl/a$, where the subscript t stands for "tooth." In the region between PQ and $M'N'$ consider an infinitesimal conductance of width dx corresponding to the quadrant of radius x . Its conductance is $dg_s = \gamma l dx / (\frac{1}{2} \pi x + a)$, where the subscript s stands for slot. Integrating between the limits $x = 0$ and $x = \frac{1}{2} s$ we find $g_s = (2 \gamma l / \pi) \ln (\pi s / 4 a + 1)$.

* This and the remaining problems in the chapter have a direct application in magnetic calculations of electro-dynamic machinery. Thus, electrode B may represent a magnetic pole-piece, and electrode A part of a slotted armature. The required resistance corresponds to the reluctance of the air-gap.

The total conductance $g = g_t + g_s$, and consequently the required resistance $R = 1/(g_t + g_s)$.

Prob. 18. — In Prob. 17 let the teeth of the electrode A be circular cylinders of diameter t . Determine the resistance of the medium corresponding to one tooth assuming the same shape of stream lines.

Prob. 19. — Two flat electrodes of width b (Fig. 5) extend indefinitely in the direction perpendicular to the plane of paper. They are

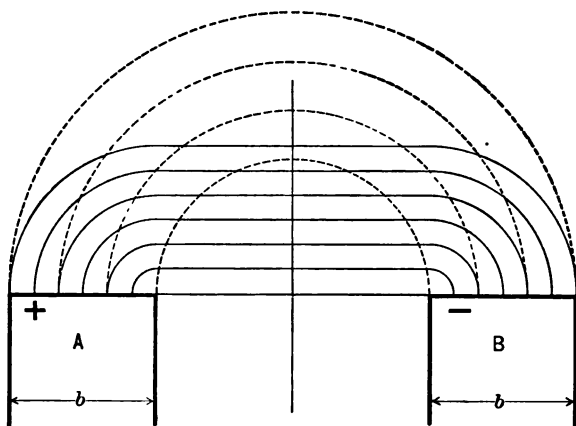


FIG. 5. — Two assumed systems of lines of flow of current between flat electrodes.

surrounded by a medium of low conductivity, and the stream lines of the current are assumed to consist of straight lines connected by concentric quadrants (full drawn lines). Deduce an expression for the conductance of the medium over a length l in the direction perpendicular to the plane of the paper.

Prob. 20. — Solve Prob. 19 assuming the lines of flow to be concentric semi-circles, as shown in Fig. 5 by dotted lines.

Prob. 21. — The stream lines between electrodes A and B (Fig. 6) have a general shape shown in the figure. The minimum distance between the electrodes is a , the maximum distance, measured along a line of flow, is $a + \Delta a$. The lower electrode is so proportioned that the increase in the length y of lines of flow is proportional to the square of the distance x from the center. Calculate the conductance of the medium for a length l in the direction perpendicular to the plane of the paper, and determine the equivalent

distance a_{eq} between two flat electrodes which would give the same conductance.* *Hint.* $y = a + \Delta a (x/w)^2$; the infinitesimal conductance is $dg = \gamma l dx/y$. The distance a_{eq} is determined from the condition $g = \gamma lw/a_{eq}$.

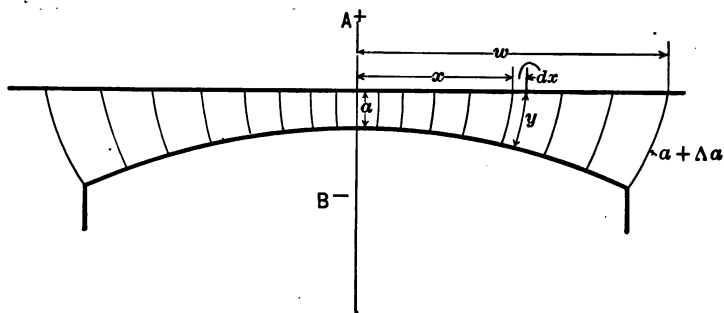


FIG. 6. — A gap of variable length between two electrodes.

Prob. 22. — Solve Prob. 21 when the increase in the length y is proportional to the n th power of the distance x from the center.

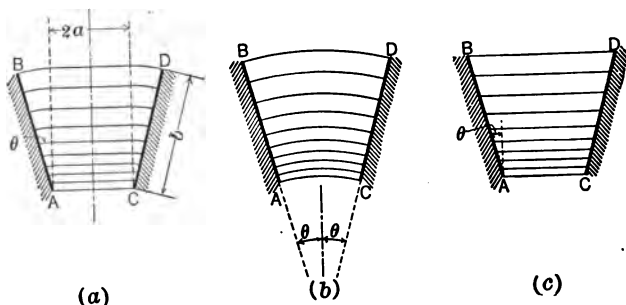


FIG. 7. — Lines of flow of current between two inclined planes; three simple assumptions.

Prob. 23. — Two electrodes (Fig. 7) have plane surfaces inclined at an angle 2θ to one another. It is required to calculate the conductance of the medium between them, assuming in succession the lines of flow to be shaped as shown in (a), (b) and (c).

* For a practical application of this problem see the author's "Magnetic Circuit," pp. 91 and 92.

Prob. 24. — In nature the actual flow of current takes place in such a way that the conductance of the path is a maximum. Show that assumption (a) in Fig. 7 is more correct than (b), and that (b) is more correct than (c). *Hint.* — Compare the conductances of the corresponding infinitesimal paths, remembering that $\tan \theta > \theta > \sin \theta$.

CHAPTER II.

MINIMUM WEIGHT OF METAL IN A COMBINATION OF CONDUCTORS.

MANY engineering problems admit of a large number of solutions, but usually only one or a few of these solutions satisfy the condition of maximum economy, or minimum expense. One of such cases is the selection of sizes of conductors in a network used for lighting or for power distribution. The total voltage drop between the generating station and the consumer is usually given, but this voltage drop may be arbitrarily divided, allowing part of it in the main feeders, another part in secondary feeders, and the remainder in the distributing conductors. If a large part of the total drop be allowed in the feeders, these feeders can be made of comparatively small cross section, but the distributing conductors would have to be comparatively large, and *vice versa*. The problem becomes definite by adding the condition that the total weight of copper (or any other metal) be a minimum. This method of determining cross sections of conductors is applied to a few practical cases in the problems below. However, the student must understand that the answer so obtained is often modified in practice for the following reasons:

(a) A cross section which gives theoretically the least amount of metal may not be a standard size, and it is usually cheaper to use a standard conductor than to order a special size drawn and insulated.

(b) Even with standard sizes, it is very desirable to reduce the number in a given installation to a minimum, so as to avoid mistakes in repairs, and not to keep too large a stock.

(c) The conductors must satisfy the condition that the current density shall not exceed a certain practical limit, owing to the danger of overheating, damage to insulation, and fire risk.

(d) Larger conductors than are necessary for electrical requirements are often used for mechanical strength, or in anticipation of the growth of the system.

Prob. 1. — Prove that one conductor of length l is cheaper than two conductors of different cross section in series. *Solution.* — Let the cross sections of two conductors be q_1 and q_2 respectively, and their lengths l_1 , and $l_2 = l - l_1$. Let the total permissible voltage drop be e , and the current i . If the unknown voltage drop in one conductor be x , that in the other must be $e - x$. Applying Eq. (3) of the preceding chapter, we get two conditions: $x = i\rho l_1/q_1$, and $e - x = i\rho (l - l_1)/q_2$. The total volume of metal must be a minimum, or $q_1 l_1 + q_2 (l - l_1) = \min$. Substituting the values of q_1 and q_2 from the first two equations into the third, we get:

$$u = l_1^2/x + (l - l_1)^2/(e - x) = \min.$$

According to the general rule of the differential calculus, the partial derivatives of this expression with respect to l_1 and x must be equated to zero. In this case both conditions lead to the same equation: $x/l_1 = (e - x)/(l - l_1)$, which means that the voltage drop per unit length must be the same in both conductors. But since the current is the same, this condition means that $q_1 = q_2$. See the note at the end of the chapter.

Prob. 2. — Solve Prob. 1 using the method of indeterminate coefficients. *Solution.* — The total voltage drop in the two conductors is $e = i\rho (l_1/q_1 + l_2/q_2)$. The sum of the lengths being given we have $l_1 + l_2 = l$. Thus, multiplying the first equation by an indeterminate coefficient λ_1 and the second equation by λ_2 , and adding to the given condition of minimum, we get: $q_1 l_1 + q_2 l_2 + \lambda_1 (l_1/q_1 + l_2/q_2) + \lambda_2 (l_1 + l_2) = \min$. In this equation, q_1 , l_1 , q_2 , l_2 can now be treated as independent variables. Equating to zero

the partial derivatives of the expression, the desired result is obtained.

Prob. 3. — Two conductors of lengths l_1 and l_2 , and cross sections q_1 and q_2 , are to be connected in parallel across a source of voltage E . The total current i drawn by the two conductors is given and also the sum of the cross sections. Prove that the weight of metal is a minimum when $l_1 = l_2$. *Hint.* — If the current through one conductor is x , and through the other $(i - x)$, we have, according to Eq. (7) in the preceding chapter: $x = E\gamma q_1/l_1$, and $i - x = E\gamma q_2/l_2$.

Prob. 4. — An electric light main (Fig. 8) has groups of lamps connected to it at different places. It is required to determine the cross sections q_1, q_2, q_3 , etc., in such a way that the total weight of

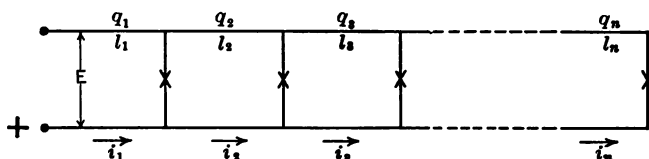


FIG. 8. — An electric street cable with house connections.

metal in the conductors be a minimum. The applied voltage E is given, as well as all the currents. The total voltage drop in the conductor must not exceed $2e$, so that the voltage across the most remote lamps is $E - 2e$. *Solution.* — The voltage drop in one of the conductors, say the positive one, is

$$e = \rho \Sigma i l / q. \quad (1)$$

The condition for a minimum volume of metal is

$$V = \Sigma l q = \min. \quad (2)$$

The problem is one in relative maxima and minima, so that we have:

$$\Sigma l q + \lambda^2 \Sigma i l / q = \min., \quad (3)$$

where λ^2 is an indeterminate coefficient. Taking partial derivatives with respect to cross sections q , we obtain n equations of the form

$$l_k - \lambda^2 i_k l_k / q_k^2 = 0,$$

or, solving for q_k ,

$$q_k = \lambda \cdot \sqrt{i_k l_k}. \quad (4)$$

Substituting these values of q 's in Eq. (1), we get:

$$\lambda = (\rho/e) \Sigma l \sqrt{i_k}. \quad (5)$$

Knowing λ , the cross sections are computed from Eq. (4).

Prob. 5. — Show that in the preceding problem the current density i_k/q_k decreases from the generator to the last consumer.

Prob. 6. — Solve Prob. 4, substituting the condition of constant current density in place of Eq. (2).

Prob. 7. — Let the currents consumed in the four groups of lamps in Fig. 8 be 40, 30, 50, and 27 amperes respectively. Let the distances be 1000, 3000, 450 and 1200 meters, and the applied voltage 228. The voltage across the furthest lamps must not be less than 220. Find the cross sections of the conductors and the total weight of copper, according to three different suppositions: (1) The weight is to be a minimum; (2) the current density is to be the same in all the sections; (3) the whole conductor is to be of a uniform cross section. For copper $\rho = 0.018$ ohm, when lengths are measured in meters and cross sections in square millimeters.

Prob. 8. — Electric power is to be transmitted from a water-fall *A* (Fig. 9) to two towns *C* and *D*. Between *A* and *B* the line

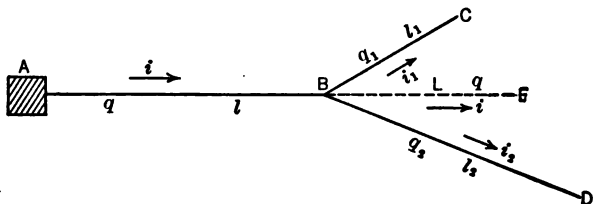


FIG. 9. — A branched transmission line.

is common to both towns. Knowing the currents i_1 and i_2 , the lengths of the lines, and the permissible voltage drop $2e$ between *A* and *C* which is equal to that between *A* and *D*, find the cross sections of the wires such that the total weight of the conductors will be a minimum. *Solution.* — The volume of the conductors is

$$ql + q_1l_1 + q_2l_2 = \min. \quad (6)$$

Let the unknown drop of voltage between *A* and *B* be x . Then we have three conditions:

$$e - x = \rho l i_1 / q_1 = \rho l_2 i_2 / q_2, \quad (7)$$

and

$$x = \rho l (i_1 + i_2) / q. \quad (8)$$

Substituting the values of q_1 , q_2 , and q from Eqs. (7) and (8) into Eq. (6) and differentiating with respect to x , we get after reduction:

$$\frac{l \sqrt{i_1 + i_2}}{x} = \frac{\sqrt{l_1^2 i_1 + l_2^2 i_2}}{l - x}. \quad (9)$$

From this equation x can be calculated, and subsequently the three unknown sizes of conductors found.

Prob. 9. — The voltage at A (Fig. 9) is 1300 volts (for electric railway service); the voltage at C and at D must not be below 1150 volts when the current consumption at C is 100 amperes, and at D 200 amperes. The distances are $l = 4$ km.; $l_1 = 2$ km.; $l_2 = 1.3$ km. Determine the most economical size of conductors and the total cost of copper at 40 cents a kilogram.

Prob. 10. — Solve the preceding problem and compare the answers when, instead of minimum weight, the cross sections satisfy one of the following two conditions: (a) the current density is the same in the three branches; (b) the cross sections of the three branches are equal, and the voltage drop to neither of the points of consumption exceeds the prescribed limit.

Prob. 11. — Show that the result obtained in Prob. 8 may be interpreted as follows: Replace conductors BC and BD by a fictitious conductor BF (Fig. 9) of the same cross section as AB , and of a length

$$L = \sqrt{l_1^2 \left(\frac{i_1}{i_1 + i_2} \right) + l_2^2 \left(\frac{i_2}{i_1 + i_2} \right)}. \quad \dots \quad (10)$$

The fictitious conductor is supposed to carry the same current ($i_1 + i_2$) as AB . The introduction of this conductor is convenient in numerical applications, because knowing L and l the voltage drop x is found from the proportion $x : e = l : (l + L)$.

Prob. 12. — Show that in the case of more than two conductors radiating from point B (Fig. 9) the length of the fictitious conductor is

$$L = \sqrt{\frac{\sum l_i^2 i_i}{\sum i_i}}, \quad \dots \quad (11)$$

the summation being extended over all the branches except AB .

Prob. 13. — Extend the solution of Prob. 8 to the case when the voltage drop to point C is different from that to D . Show that x must be determined from an equation of the fourth degree, and indicate how it may be solved by trial.

Prob. 14. — Referring to Fig. 9, let there be n branches connected to AB at point B , and let a different voltage drop be prescribed for each branch. Show that the condition for a minimum weight of copper is

$$\frac{q^2}{\sum i} = \sum_{k=1}^{k=n} \frac{q_k^2}{i_k}. \quad \dots \quad (12)$$

Prob. 15. — Show how to determine the sizes of the conductors in the branched transmission system shown in Fig. 10, to satisfy the condition of minimum copper or aluminum. The total drop of voltage to the five substations *D*, *E*, *G*, *H*, and *K* must be the same. *Solution.* — Replace the three conductors 6, 7, 8 by an

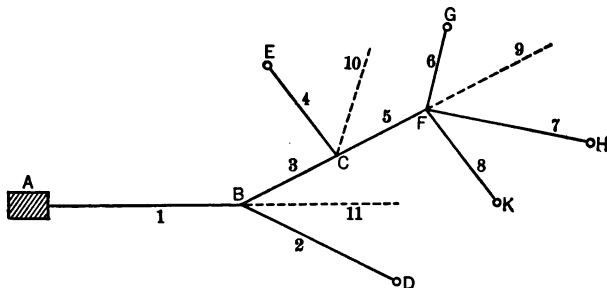


FIG. 10. — A branched transmission system.

equivalent conductor 9, according to formula (11). Then add the length of 9 to that of 5, and replace the two conductors 4 and (5 + 9) by an equivalent conductor 10. Similarly, replace conductors (10 + 3) and 2 by the fictitious conductor 11, so that the given network of conductors is finally replaced by one simple conductor (1 + 11) carrying the total current Σi . The cross section of this conductor is found from the total given voltage drop, so that the voltage drop in conductor 1 can be calculated, and the cross sections of conductors 1 and 2 found. The remainder of the system is then treated in the same way, using the rest of the prescribed drop of voltage.

NOTE. — In problems in which partial derivatives of a function are equated to zero in order to find a minimum value, strictly speaking, it is necessary to investigate whether or not the values of the variables so found actually give a minimum. This can be done by investigating the sign of a small increment of the function, at and near the critical values of the variables. If the values of the variables actually correspond to a minimum, then this increment is positive for any small variations of the variables whatsoever, positive or negative. In practice, this test is often omitted, because usually one can satisfy oneself by a physical interpretation of the problem that no other stationary value is possible, except a true minimum. The same remark applies of course to a maximum.

However, should there be any doubt, such a test ought to be performed. For instance, in the case of Probs. 1 and 2 we have

$$l_1/q_1 + l_2/q_2 = a = \text{Const.}; \quad \dots \quad (13)$$

$$l_1 + l_2 = l; \quad \dots \quad (14)$$

$$u = l_1 q_1 + l_2 q_2. \quad \dots \quad (15)$$

Solving the first two equations for l_1 and l_2 , and substituting their values into the third equation, we get

$$u = l (q_1 + q_2) - a q_1 q_2. \quad \dots \quad (16)$$

Let δq_1 and δq_2 be some small increments of q_1 and q_2 . Then, according to Taylor's theorem for a function of two variables, we have

$$\delta u = (\delta q_1 D_1 u + \delta q_2 D_2 u) + \frac{1}{2} (\delta q_1^2 D_1^2 u + 2 \delta q_1 \delta q_2 D_1 D_2 u + \delta q_2^2 D_2^2 u) + \text{etc.}$$

Taking partial derivatives of u , we get

$$D_1 u = l - a q_2; \quad D_2 u = l - a q_1; \quad D_1^2 u = 0; \quad D_2^2 u = 0; \quad D_1 D_2 u = -a.$$

Consequently,

$$\delta u = (l - a q_2) \delta q_1 + (l - a q_1) \delta q_2 - a \delta q_1 \delta q_2.$$

For the critical or stationary value of u the coefficients of δq_1 and δq_2 must be equal to zero, and we get the result already obtained, namely,

$$q_1 = q_2 = l/a, \quad \dots \quad (17)$$

so that

$$\delta u = -a \delta q_1 \delta q_2. \quad \dots \quad (18)$$

Thus, near the stationary value, the sign of δu depends upon whether δq_1 and δq_2 are of the same sign or of opposite signs. This can be determined from Eq. (13) by taking a complete variation:

$$-(l_1 \delta q_1 / q_1^2 + l_2 \delta q_2 / q_2^2) + (\delta l_1 / q_1 + \delta l_2 / q_2) = 0. \quad \dots \quad (19)$$

At the critical value of the function, $q_1 = q_2$, and, according to Eq. (14), $\delta l_1 + \delta l_2 = 0$. Therefore, in Eq. (19) the expressions in both parentheses are equal to zero separately. The expression in the first parenthesis gives

$$\delta q_1 / \delta q_2 = -l_2 q_1^2 / (l_1 q_2^2). \quad \dots \quad (20)$$

Thus, when l_1 and l_2 are positive, as is presupposed in this particular physical problem, δq_1 and δq_2 are of opposite signs. Therefore, in Eq. (18) δu is always positive, and the values of q_1 and q_2 according to Eq. (17) actually correspond to the minimum value of the function.

CHAPTER III.

MOST ECONOMICAL SIZE OF CONDUCTORS.

IN the preceding chapter the loss of voltage in the conductors is given, and therefore the waste of power in the form of i^2r heat in the distributing network is fixed. Consequently, the most economical sizes of conductors are those which give a minimum weight of copper for the whole installation. Such conditions usually obtain in comparatively small local distributing systems in which the voltage drop is not more than a few per cent of the applied voltage.

In large transmission systems the permissible drop of voltage, or, to be more accurate, the permissible loss of power in the line, is much greater and may vary within comparatively wide limits. In this case the proper selection of the size of conductors depends upon two factors: the cost of the line itself, and the cost of power lost in the line as Joulean heat i^2r . The following table will make this clear:

1	2	3	4	5	6
Size of wire, mm. ² .	Cost of wire, dollars.	Ten per cent for interest, sinking fund and upkeep, dollars.	Line loss, kilowatt-hours per year.	Annual cost of power lost in the line, dollars.	Total operating cost of transmission, dollars. (3) + (5)
120	37,500	3750	136,000	1633	5383
90	28,125	2812	181,300	2178	4990
80	25,000	2500	204,000	2450	4950
70	21,875	2187	233,100	2800	4987
50	15,625	1562	326,400	3920	5482

A certain amount of power is to be delivered at a considerable distance from the power house, by means of a high-tension transmission line. Different standard sizes of conductors which may be considered for this transmission are given in the first column in the table. The corresponding total cost of conductors is given in the second column. The third column gives the amount to be written off annually from the gross income, in order to pay for the interest, sinking fund, and repairs on the line. This item is estimated in this case to amount to about 10 per cent of the cost of the wire itself. The fourth column gives the values of the heat loss, i^2r , in the line; the larger the size of the wire the smaller this loss, being directly proportional to the resistance of the wire, in other words, inversely proportional to its cross section. In the fifth column will be found the actual cost of this line loss, in so far as the generation of power is concerned, or the loss of operating income. This item must also be subtracted from the gross income derived from the power sold.

Since the gross income may be considered fixed, the most economical size of conductor is that for which the total expense (column 6) is a minimum; in our case 80 mm.². By taking a larger size of wire, say 120 mm.², the loss in the line is reduced from \$2450 to \$1633, but the interest and depreciation increase from \$2500 to \$3750, so that the total cost of transmission is increased from \$4950 to \$5383. Reducing the size of wire to 50 mm.² is not advisable either; the interest and depreciation in this case are comparatively small, only \$1562, but the cost of power lost in the line and the total cost of transmission are again higher.

The table shows that the total cost is a minimum,

when the items in columns 3 and 5 are approximately equal. It is proved below that this is the necessary theoretical condition for a minimum operating cost. The table shows also that it is permissible in practice to deviate considerably from the most economical cross section, without appreciably increasing the cost of transmission. This fact is illustrated in Fig. 11: while

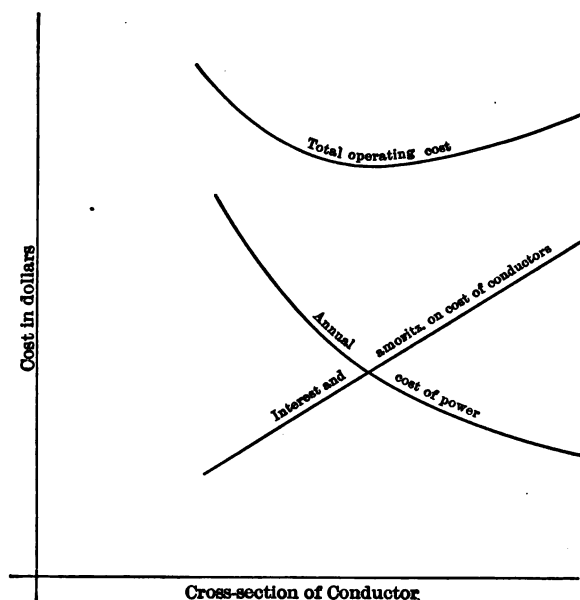


FIG. 11. — Items in the operating cost of power transmission.

the slope of both component curves is considerable, the resultant curve is quite flat near the point of minimum. Therefore, it is permissible in practice to modify to a considerable extent the theoretically most economical cross section in order to satisfy some other important conditions. For instance, if the capital necessary to start the enterprise is limited, it may be

advisable to choose the 70 mm.² conductor instead of the 80 mm.². On the contrary, if the capital is plentiful, and the demand for power may increase in the near future, it is better to take 90 mm.² conductors.

The most economical cross section is not always the one for which the total cost of transmission is a minimum. It will be seen from the problems below that sometimes it is proper to design the line for a maximum of net income; again, in other cases, the line is to be so proportioned as to insure the greatest percentage gain on the invested capital. The student should not expect that the solution of the problems given below would enable him to handle practical transmission projects in their entire complexity. These problems are intended only to elucidate an elementary economic aspect of the subject that can be reduced to mathematical maxima and minima.

The total operating expense always includes large items independent of the size of conductors, for instance, the interest on the cost of transmission towers, the cost of patrolling the line, etc. (see Prob. 2). Therefore, the total operating expense is a function of the cross section q of the conductors plus a constant quantity. When taking the derivative of the function in order to obtain a minimum, the constant term vanishes, so that the desired value of q is independent of the constant items of the operating cost.

In order to simplify the mathematics and to avoid going into purely electrical matters, all problems are solved for direct current, and not for alternating current transmission. With certain limitations, the results hold true for the latter, especially at power factor values near 100 per cent.

Prob. 1. — In the table given above the cost C of wire may be represented by the formula

$$C = aq, \quad (1)$$

where q is the cross section of the wire and a is a constant depending upon the length of the line and the cost of wire per kilogram. The annual cost W of energy lost in the line can be expressed by the formula

$$W = b/q, \quad (2)$$

where b is the annual expense when the cross section of the wire equals 1 mm.². Prove that the most economical size of the wire is the one for which $pC = W$, where p is the fraction of the original cost allowed for the interest, sinking fund, and upkeep of the wire per year. This relationship is known as Kelvin's law of economy.

Solution. — The total cost of transmission per year is

$$X = pC + W = paq + b/q. \quad (3)$$

Equating to zero the first derivative with respect to q , we get $pa - b/q^2 = 0$, or

$$q = \sqrt{b/pa}. \quad (4)$$

Substituting this value in Eqs. (1) and (2), the desired relation is obtained.

In Eq. (3) it is required to find the minimum value of the sum of two variables paq and b/q . The product of these variables is a constant quantity, pab , so that Theorem 1 in the Appendix applies, and we can write directly that

$$paq = b/q, \quad (5)$$

or $pC = W$. From Eq. (5) the value of q given in Eq. (4) follows directly.

Prob. 2. — In designing a transmission line it is found that the total annual cost X of transmission is the following function of the cross section q of the conductors:

$$X = A + Bq^m + C/q^n,$$

where A , B , C , m , and n are known constants. Determine the most economical value of q .

Prob. 3. — A certain amount of power is to be transmitted by means of a high-tension direct-current line from a waterfall to a substation in a large city. Outside the city limits, the transmission is to be by means of bare wires on towers; within the city limits

it must be by means of underground cables. Determine the size of the bare wire and of the cable conductors under the condition that total annual expense for the interest, upkeep, and depreciation of the line be a minimum. Total voltage drop between the power house and the substation is fixed. *Solution.* — Let the length and the cross section of the conductors for the overhead transmission be l_1 and q_1 , those for the cable l_2 and q_2 . Let the corresponding drops of voltage be e_1 and e_2 . Then, according to Eq. (3), Chapter I,

$$q_1 = I\rho l_1/e_1; \quad q_2 = I\rho l_2/e_2, \quad \dots \dots \dots (6)$$

and the condition must be fulfilled that

$$e_1 + e_2 = \text{Const.} \quad \dots \dots \dots (7)$$

Let the cost of the conductors per unit length and per unit cross section be a_1 and a_2 respectively. Then the annual cost of these conductors must satisfy the condition

$$p_1 a_1 l_1 q_1 + p_2 a_2 l_2 q_2 = \text{min.}, \quad \dots \dots \dots (8)$$

where p_1 and p_2 are the fractions of the original cost to be allowed as the annual expense. Substituting the values of q_1 and q_2 from Eq. (6) and omitting the common factor, ρI , we get

$$p_1 a_1 l_1^2/e_1 + p_2 a_2 l_2^2/e_2 = \text{min.} \quad \dots \dots \dots (9)$$

The total differential of this expression must be equal to zero, or

$$-p_1 a_1 l_1^2 de_1/e_1^2 - p_2 a_2 l_2^2 de_2/e_2^2 = 0.$$

But from Eq. (7), $de_1 + de_2 = 0$, so that, eliminating de_1 and de_2 from the preceding equation, we finally get

$$\frac{e_1}{e_2} = \frac{l_1}{l_2} \sqrt{\frac{p_1 a_1}{p_2 a_2}}. \quad \dots \dots \dots (10)$$

Eq. (10) shows the ratio in which the total given voltage drop must be divided between the overhead and the underground conductors in order to reduce the expense to a minimum. Knowing e_1 and e_2 , the corresponding values of q_1 and q_2 are calculated from Eqs. (6).

Prob. 4. — Power is to be transmitted to a city by bare overhead conductors for a distance of 10 km. and by underground cables within the city limits for a distance of 3 km. The total permissible voltage drop is 10 per cent of the generator voltage. What is the most economical per cent drop in the overhead line and in the cable if the unit price for the cable is 2.5 times that for the overhead conductor? Interest and amortization for the cable is estimated

at 6 per cent, while for the bare conductor 10 per cent should be allowed, due to its exposure.

Prob. 5. — Extend the solution of Prob. 3 to the case of a line consisting of n sections in series, all having different given values of a , p and l . *Solution.* — Eqs. (6) are of the same form, only there are n of them. Eq. (7) becomes

$$\sum_{k=1}^{k=n} e_k = e_1 + e_2 + \dots + e_n = \text{Const.} \quad (7')$$

Eq. (8) becomes

$$\sum_{k=1}^{k=n} p_k a_k l_k q_k = \text{min.}, \quad (8')$$

or, using Eqs. (6),

$$\sum_{k=1}^{k=n} p_k a_k l_k^2 / e_k = \text{min.} \quad (9')$$

Eqs. (7') and (9') are solved by the general method of relative maxima and minima. Multiply Eq. (7') by an indeterminate coefficient λ and add to Eq. (9'). Omitting the constant term, the result is:

$$\sum_{k=1}^{k=n} (p_k a_k l_k^2 / e_k + \lambda e_k) = \text{min.}$$

In this equation all e 's are independent variables, so that taking partial derivatives we obtain n equations of the general form $p_k a_k l_k^2 / e_k^2 + \lambda = 0$. Consequently

$$e_1 : e_2 : \dots : e_{n-1} : e_n = l_1 \sqrt{p_1 a_1} : l_2 \sqrt{p_2 a_2} : l_3 \sqrt{p_3 a_3} : \text{etc.}$$

This shows the ratio in which the total voltage drop must be divided among the different sections of the line, in order to have a minimum annual expense in interest, upkeep, and depreciation.

Prob. 6. — Determine the most economical current density in a transmission line, taking into account the extra interest and amortization on the cost of the power plant. *Solution.* — The annual expenses to be considered in this case consist of three parts: (1) Interest and depreciation on the conductors. (2) Interest and depreciation on the excess equipment in the power house necessary to cover the losses in the conductors. (3) The cost of generation of extra energy to cover the loss in the line.

Let the current to be transmitted be i amperes, and let the voltage drop in the line be e . Then we have as before

$$e = il\rho/q, \quad \dots \dots \dots (11)$$

so that the equipment in the power house must generate ei extra watts to cover the line loss. If the equipment is in operation T hours per year, the total annual expense X amounts to

$$X = p_1 aql + p_2 bei + cTei = \min. \quad \dots \dots \dots (12)$$

In this equation the term $p_1 aql$ represents the interest and depreciation at the rate of 100 p_1 per cent on the cost aql of the line. The second term, $p_2 bei$, represents the interest and depreciation at a rate of 100 p_2 per cent on the extra rating of the machinery necessary to generate ei watts loss in the line, b being the cost of this machinery per watt of rating. The last term, $cTei$, represents the cost of the energy lost in the line, at the rate of c dollars per watt-hour.

Since the most economical current density δ in the line conductors is required, the variables q and e in Eq. (12) must be expressed through this density

$$\delta = i/q. \quad \dots \dots \dots (13)$$

It follows from Eq. (11) that $e = l\rho\delta$, and dividing both sides of Eq. (12) by a constant quantity ρil , we finally obtain

$$X/(\rho il) = (p_1 a/\rho) (1/\delta) + (p_2 b + cT) \delta = \min. \quad \dots (13a)$$

To find a minimum of cost, it is necessary to take the derivative of this expression with respect to δ , and to equate it to zero. Noticing, however, that Theorem 1 in the Appendix applies in this case, we have $(p_1 a/\rho) (1/\delta) = (p_2 b + cT) \delta$, from which

$$\delta = \frac{\sqrt{p_1 a/\rho}}{\sqrt{p_2 b + cT}}. \quad \dots \dots \dots (14)$$

This formula has been given an interesting interpretation by Hohenegg.* The quantity

$$z_1 = \sqrt{p_1 a/\rho}, \quad \dots \dots \dots (15)$$

which enters into the numerator of the expression for δ , depends only upon the transmission line constants. Hohenegg calls z_1 the "conductor factor" (Leitungszahl). The denominator,

$$z_2 = \sqrt{p_2 b + cT}, \quad \dots \dots \dots (16)$$

* C. Hohenegg, *Anordnung und Bemessung elektrischer Leitungen*, Chap. III, p. 68.

depends only upon the operating conditions in the power house. Hochenegg gave it the name of "operating coefficient" (Betriebszahl). Thus, we get the simple relation that the most economical current density is equal to the ratio of the conductor factor to the operating coefficient:

$$\delta = z_1/z_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Prob. 7. — Determine the most economical current density in the line conductors for a power transmission system operating at full load for 3500 hours a year, the cost of generation of energy being 0.5 cent per kilowatt-hour (= 1000 watt hours). Copper conductors are estimated to cost 360 dollars per metric ton (= 1000 kilograms), and the interest and depreciation on the conductors is assumed to be at a rate of 9 per cent per year. The excess capacity of the power house equipment costs 40 dollars per kilowatt of rating, and its interest and depreciation is at a rate of 10 per cent per year.

Prob. 8. — At a waterfall a certain limited amount of energy is available in the form of a current i at E kilovolts. The cost of generation is c_1 cents per kilowatt-hour. The power is to be transmitted to a distance of $\frac{1}{2}l$ km., to be sold there at a rate of c_2 cents per kilowatt-hour. Determine the size of the line conductors such that the difference between the selling price of the energy delivered and the total cost of generation and transmission will be a maximum. *Solution.* — Let the unknown cross section of the line be q ; then the voltage drop in the line, in kilovolts, is

$$e = il\rho/q, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and the sum to be realized annually from the sale of the energy is $c_2 i (E - il\rho/q) T$, where T is the number of hours per year during which the energy is used. The cost of generation per hour is $c_1 i E$, and the interest and depreciation on the cost of the line conductors is $paql$. The condition to be satisfied is that

$$c_2 i (E - il\rho/q) T - c_1 i E T - paql = \max. \quad . \quad . \quad (19)$$

Omitting the constant terms, we find that the sum $c_2 i^2 l \rho T / q + paql$ must be a minimum. Since the product of these terms is a constant, the condition for the minimum is $c_2 i^2 l \rho T / q = paql$, or

$$q = i \sqrt{\frac{c_2 \rho T}{pa}}. \quad . \quad . \quad . \quad (19a)$$

The corresponding current density is

$$\delta = \frac{i}{q} = \frac{\sqrt{pa/\rho}}{\sqrt{c_2 T}}. \quad . \quad . \quad . \quad (19b)$$

This expression is similar to Eq. (14).

Prob. 9. — Solve the preceding problem when the cross section of the wire should be such as to yield the greatest possible per cent income on the capital aql invested in line conductors. Interpret the result in commercial terms. *Hint.* — Divide the left-hand side of expression (19) by aql and determine the value of q which makes it a maximum.

Prob. 10. — Illustrate the difference between the conditions imposed in Probs. 8 and 9 on some numerical example, for instance, using the following data. The available power is 2000 kilowatts; it is to be transmitted at 22,000 volts to a town distant 20 km. The power is to be used 3000 hours a year, and to be sold at 5 cents per kilowatt-hour. The cost of one kilowatt-hour at the generating end of the line is 1.5 cents. For other data see Prob. 7.

Prob. 11. — As a modification of Prob. 8, let voltage E at the power house be unknown, but voltage E' at the receiving end be given. The total amount of power available at the waterfall is limited to P kilowatts. Find the most economical size of the conductors. *Hint.* — Eq. (19) becomes $c_2 i E' T - c_1 i (E' + e) T - paql = \max$. It contains three variables: i , e , and q . Two of these can be eliminated by means of Eq. (18) and the relation $P = (E' + e) i$, or else the method of indeterminate coefficients can be used.

Prob. 12. — Prove that the greatest amount of power which it is possible to deliver over a given line and at a given generator voltage is obtained when one half of the generated power is lost in the line; in other words, when the efficiency of the line is 50 per cent. Illustrate this result by means of curves. *Solution.* — Let the values of the current and the voltage at the generator end be i and E ; let the resistance of the line be r ohms. Then the delivered power is $P = (E - ir) i = Ei - i^2 r$. This expression is a maximum when $E = 2ir$, or the voltage drop in the line is $ir = \frac{1}{2} E$. To current as abscissæ plot curves of generated power, of power lost in the line, and of the delivered power. Take the current corresponding to the maximum power as 100 amp., the generator voltage 100 kilovolts; the resistance of the line must then be 500 ohms.

Prob. 13. — A power transmission company considers delivering electric power to a city, over an existing transmission line of resistance r . The receiver voltage E' is fixed. What amount of power would give a maximum net revenue to the company? *Solution.* — Two cases have to be considered, according to whether or not the company can elsewhere dispose of the power not sold to the city.

(1) There is a ready market for all the power available. Let i be the value of the current, c_1 the cost of generating one kilowatt-hour, c_2 the selling price of a kilowatt-hour. Then the condition for maximum net revenue is expressed as follows:

$$c_2 E' i - (c_1 E' i + c_2 i^2 r) = \max. \quad (20)$$

In this expression E' is in kilovolts, i is in amperes, and r is in kilo-ohms. The first term represents the income, the second term the cost of the energy generated and transmitted, and the third term represents the loss in net revenue due to the loss of energy in the line; all three terms are in cents or in dollars per hour. Since by supposition there is a ready market for all the energy generated, the line loss is charged at the selling rate c_2 . Taking a derivative with respect to i and equating it to zero, an expression is obtained from which i can be calculated.

(2) The company cannot readily dispose of the excess power. In this case the loss in the line cannot be charged at the rate c_2 , but only at the actual cost c_1 . In other respects Eq. (20) remains the same.

Prob. 14. — Apply the theory given in Prob. 13 to the following case: Energy is to be delivered at 33,000 volts over a line having a resistance of 27 ohms. Energy can be sold wholesale at 4 cents per kilowatt-hour, while the total cost of generation is 2.1 cents per kilowatt-hour. Find the most advantageous load in kilowatts to be transmitted under the two suppositions considered above. Plot curves showing the amount of gross and net revenue and the loss for various amounts of power transmitted.

Prob. 15. — A number of villages are situated approximately along the circumference of a circle (Fig. 12); it is planned to supply them with electric power from a power house A situated at the center of the circle. Power is to be furnished by means of a circular distributing line connected to radial feeders. The consumers and the current consumption may be considered to be uniformly

distributed over the ring conductor. The problem is to find the number n of feeders and the cross sections of the feeders and of the distributing line such that the total cost of installation will be a minimum. The voltage drop both in the feeders and in the

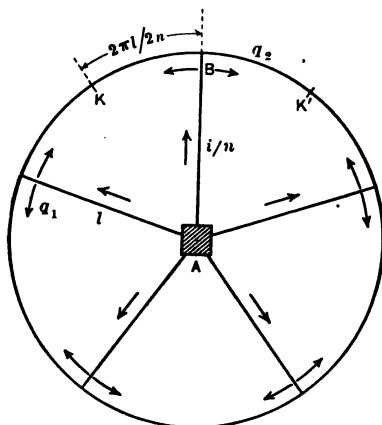


FIG. 12. — A centrally located power house with a number of consumers along the circle.

distributing conductors is prescribed. *Solution.* — Let the most economical number of feeders be n and let the prescribed voltage drop in each be e_1 . Let the total given current consumption be i , so that the current per feeder is i/n , and its cross section is

$$q_1 = \frac{(i/n) \cdot l \rho}{e_1} \dots \dots \dots (21)$$

The cost of each feeder per conductor is

$$C_f = l (a_1 q_1 + b_1) + c_1, \dots \dots \dots (22)$$

where a_1 , b_1 , and c_1 are certain constants. This expression is more complicated than those assumed in the preceding problems, because each new feeder means a new pole line, and extra switching and protective apparatus at both ends. Therefore, the total cost of a feeder line consists of three items: $l a_1 q_1$, which is the cost of the conductor itself; $l b_1$, which is the cost of the pole line, independent of the cross section of the conductor; and c_1 , which represents the cost of the switching and protective apparatus, etc., at both ends of the feeder. Each feeder, such as AB , supplies two sections, BK and

BK' , of the distributing line. The uniformly distributed load between B and K may be considered concentrated in the middle between B and K . Therefore, the cross section of line KB is

$$q_2 = \frac{(i/2n) \cdot (2\pi l/2n) \rho}{e_2}, \quad \dots \quad (23)$$

and the cost of the distributing line over the whole circumference, per conductor, is

$$C_d = 2\pi l \cdot a_2 q_2. \quad \dots \quad (24)$$

The terms independent of the cross section, such as b_1 and c_1 , do not have to be considered here, because they vanish in the differentiation with respect to n .

The total cost of installation is $C_t = nC_f + C_d$, or

$$C_t = n[l(a_1 q_1 + b_1) + c_1] + 2\pi l a_2 q_2. \quad \dots \quad (25)$$

Substituting the values of q_1 and q_2 from Eqs. (21) and (23) and equating to zero the first derivative of C_t with respect to n , the most economical number of feeders is determined.

Prob. 16. — A high-tension transmission line AB (Fig. 13) runs side by side with a low-tension distributing line CD , which supplies

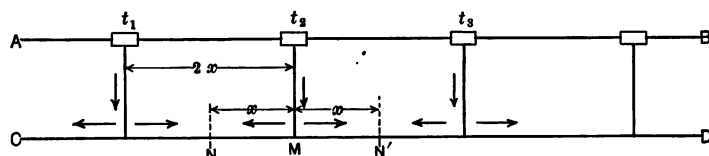


FIG. 13. — A distributing line connected to a transmission line through transformers.

electric power to uniformly distributed consumers at an average rate of w watts per meter of its length. The low-tension line is connected to the high-tension line through transformers t_1 , t_2 , t_3 , etc. The power loss per unit length of the line must not exceed a given small fraction α of the delivered power. The cost of the high-tension line being fixed, it is required to find the best distance between the transformers, their size, and the cross section of the low-tension conductors, such as to make the total cost of the installation a minimum. *Solution.* — Let the unknown distance between the transformers be $2x$ meters; then, neglecting a small loss, the size of each transformer is $2wx$ watts, and its cost $2a_1wx + b_1$, where b_1 is a term which indicates that the price of a transformer is

not exactly proportional to its rating. The cost of the transformers per meter length of the line is $a_1 w + (b_1/2x)$. Each transformer, such as t_2 , supplies power to the parts MN and MN' of the line. The power demand on the section MN of the line is xw , and the loss in the line must not be over αxw . With a uniformly distributed load, the total power consumption may be assumed to be concentrated at one third of the length of the conductor; see Prob. 17 below.* Hence, for two conductors,

$$2 i^2 r = 2 i^2 \cdot \frac{1}{3} x \rho / q = \alpha x w. \quad (26)$$

Knowing the voltage E of the distributing line, current i can be calculated because $xw = Ei$, so that Eq. (26) gives

$$q = \frac{2 x^2 w \rho}{3 \alpha E^2}. \quad (27)$$

The cost of the two conductors of the distributing line, per meter of its length, is $2 a_2 q$, so that the total cost of the installation, per meter of its length, is

$$C_t = a_1 w + \frac{b_1}{2x} + \frac{2 a_2 \cdot 2 x^2 w \rho}{3 \alpha E^2} = \min. \quad (28)$$

Taking the first derivative with respect to x and equating it to zero, gives:

$$-\frac{b_1}{2x^2} + \frac{8 a_2 w \rho x}{3 \alpha E^2} = 0, \quad (29)$$

whence the most economical distance between the transformers is

$$x = \frac{1}{2} \sqrt{\frac{3 \alpha b_1 E^2}{2 a_2 w \rho}}. \quad (30)$$

The student may well analyze this formula from an engineering point of view, making clear to himself the practical significance of the factors which enter into it, and of those which dropped out in the differentiation. It is also well worth while to multiply Eq. (29) by x and then compare it with Eq. (28). It will be seen that the total cost is a minimum when the distributing line costs twice as much as the excess cost $b_1/2x$ of the transformers. This result could be directly foreseen from Theorem 2 in the Appendix.

* A uniformly distributed electric load is concentrated in the middle of the line for the purpose of determining the voltage drop, as in Prob. 15, and is concentrated at one third the distance from the supply end for the purpose of determining the power loss, as in this problem. In the first case it is a matter of integrating a linear function ir of the current, in the second case a quadratic function $i^2 r$ of the current.

Prob. 17. — A distributing line furnishes a total current i which is uniformly consumed over its length l . Prove that the power loss in the line, due to Joulean heat, is the same as if the total current consumption were concentrated at a distance $\frac{1}{3}l$ from the generating end. *Solution.* — At a distance x measured from the end of the line remotest from the generator, the current is ix/l ; the power loss over an infinitesimal element dx of the line is $(ix/l)^2 \rho dx/q$. Integrating this expression between 0 and l we get $i^2 \cdot \frac{1}{3} l \rho/q$, which proves the proposition. The curve of power loss per unit length is a parabola, so that the result could have been foreseen from a well-known theorem concerning the position of the center of gravity of an area bound by a parabola and the axis of abscissæ. The proposition proved in this problem is used in the solution of the preceding problem.

CHAPTER IV.

A LEAKY DIRECT-CURRENT LINE.

CONSIDER a long telegraph line with a battery of voltage E_1 applied at the sending end. The insulation of the line being imperfect, there is some leakage of current over the insulators and poles to the ground. Consequently, some current is drawn from the battery even when the receiving end of the line is open. At each pole this leakage current depends upon the voltage between that particular point of the line and the ground. Since this voltage decreases from the sending to the receiving end of the line, the leakage current per pole also decreases in the same way. The current which the wire carries at a particular point of the line is equal to the sum of the leakage currents between that point and the receiving end. Thus the line current decreases from a maximum near the battery to zero at the opposite end of the line.

When the receiver circuit is closed, the total current in the line consists of two parts: the working current, which finally flows through the receiver relay, and the leakage current, which never reaches the other end. Under some conditions of bad weather and poor insulation, the leakage current of a long line may be much greater than the working current.

Strictly speaking, the line current decreases in steps, from pole to pole. In order, however, to be able to apply the calculus, it is much easier to consider the leakage uniformly distributed over the line. With a

very long line, and a great many poles, the results so obtained are very close to the actual conditions. In the case of a leaky underground cable, the assumption may become perfectly correct.

Let the resistance of the conductor be r ohms per kilometer, and let the conductance of the path between the line and the ground be g mhos (ohm^{-1}) per kilometer. With sufficient given conditions at the receiver end, or at the sending end, or both, it is required to find the law according to which the current and the voltage vary along the line.

For an infinitesimal length ds of the line, at a distance s from the receiver end, we may write Ohm's law as follows:

$$dE = ir \, ds, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where E is the voltage at that point between the line and the ground. For the leakage current over the element ds we have

$$di = Eg \, ds. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The mathematical problem, therefore, is to solve Eqs. (1) and (2) so as to find E and i as functions of s . From Eq. (2), $E = (1/g) (di/ds)$; taking a derivative of both sides of this expression we get: $dE/ds = (1/g) (d^2i/ds^2)$. But from Eq. (1), $dE/ds = ir$, so that

$$\frac{d^2i}{ds^2} = gri. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is the differential equation for the current i . In a similar manner, we obtain for the voltage E :

$$\frac{d^2E}{ds^2} = grE. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For the purposes of solution of these equations, and

for numerical computations, it is convenient to introduce a factor

$$m = \sqrt{gr}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

so that the foregoing equations become

$$\frac{d^2 i}{ds^2} = m^2 i; \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{d^2 E}{ds^2} = m^2 E. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In other words, i and E are such functions of s that their second derivatives are proportional to the functions themselves. By a simple substitution the student will find that the exponential functions e^{ms} and e^{-ms} satisfy these equations, so that the most general solution is

$$i = Ae^{ms} + Be^{-ms}; \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$E = Ce^{ms} + De^{-ms}; \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where A , B , C and D are constants of integration to be determined from the given terminal conditions of the problem. For example, let it be desired to introduce the current I_2 and the voltage E_2 at the receiver end. Then for $s = 0$ we have $i = I_2$ and $E = E_2$; and Eqs. (8) and (9) applied to these points give two equations connecting the constants of integration. Two more equations are obtained by substituting the values of i and E from Eqs. (8) and (9) into either Eq. (1) or Eq. (2).

While it is quite possible to solve the problem as outlined above, and to get the final result, both the steps and the final formulæ are somewhat involved and unsymmetrical; it is preferable to write the solution of Eqs. (6) and (7) in the form of hyperbolic func-

tions. It will readily be seen that both $\cosh ms$ and $\sinh ms$ satisfy these equations as partial solutions. Therefore the complete integrals of these equations are:

$$i = I_2 \cosh ms + F \sinh ms; \quad . \quad . \quad . \quad (10)$$

$$E = E_2 \cosh ms + K \sinh ms; \quad . \quad . \quad . \quad (11)$$

where F and K are constants of integration. The other two constants are denoted by I_2 and E_2 , because, by supposition, $i = I_2$ when $s = 0$, and $E = E_2$ when $s = 0$. Substituting the values of i and E from these equations into Eq. (1) we get

$$mE_2 \sinh ms + mK \cosh ms = rI_2 \cosh ms + rF \sinh ms. \quad (12)$$

Since this equation must hold true for all values of s , it must be an identity, so that

$$F = mE_2/r, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$K = rI_2/m. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

It is convenient to introduce a fictitious resistance

$$R = \sqrt{r/g}, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

so that Eqs. (13) and (14) become

$$F = E_2/R. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$K = RI_2. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Eqs. (10) and (11) may now be written in the following form:

$$i = I_2 \cosh ms + (E_2/R) \sinh ms, \quad . \quad . \quad (18)$$

$$E = E_2 \cosh ms + RI_2 \sinh ms. \quad . \quad . \quad . \quad (19)$$

Several other expressions may be written for i and E , depending upon the particular problem in hand, and upon whether the battery voltage is given. In all cases it is a question of properly selecting the form

of the hyperbolic functions and the constants of integration. Tables of hyperbolic functions are available in a number of books, and computations are quite simple.

It is worthy of note that were the signs on one side of Eqs. (6) and (7) reversed, the solution would be in terms of ordinary circular sine and cosine. Such is the case in simple mechanical and electrical oscillations.

Prob. 1. — The resistance of a telegraph line is 7.2 ohms per kilometer, the insulation resistance to the ground is 0.2 megohm per km.; the length of the line is 450 km. The relay used at the receiving end has a resistance of 280 ohms, and requires 0.11 amp. to operate it. Plot curves of E and i for the line from the receiving to the sending end. *Hint.* $E_2 = 280 \times 0.11 = 30.8$ volts.

Prob. 2. — Show that expressions for F and K which are identical with Eqs. (16) and (17) are obtained by substituting the values of i and E from Eqs. (10) and (11) into Eq. (2), instead of into Eq. (1).

Prob. 3. — Deduce expressions corresponding to Eqs. (18) and (19), using exponential in place of hyperbolic functions.

Prob. 4. — In order to determine the unknown conductance g of insulation, a telegraph line is opened at the receiver end, and simultaneous readings of voltages are taken at both ends. Show how to determine g from these readings, knowing the resistance r of the conductor.

Prob. 5. — In order to determine the unknown conductance g of insulation, a telegraph line is short-circuited at the receiver end and simultaneous readings of currents are taken at both ends. Show how to determine g from these readings, knowing the resistance r of the conductor.

Prob. 6. — When distances s are measured from the sending end of the line, show that the sign on one side of Eqs. (1) and (2) must be reversed, while Eqs. (6) and (7) remain the same.

Prob. 7. — Deduce expressions similar to Eqs. (18) and (19), in which s is measured from the sending end, and the constants of integration are expressed in terms of the current I_1 and the voltage E_1 at the sending end.

Prob. 8. — When the leakage conductance g is very small, m is also small, and R is very large. In Eqs. (18) and (19) replace

the hyperbolic functions by their expansions into infinite series, and show how to compute the voltage and the current at the sending end, neglecting all but two or three terms of the series. Consider the limiting case of $g = 0$, and show that the result could have been foreseen directly.

Prob. 9. — Eq. (18) may be written in the form $i = (I_2 / \cosh \theta) \cosh (ms + \theta)$, where the hyperbolic angle θ is determined from the relationship $\tanh \theta = E_2 / (RI_2)$. Write a corresponding transformation for Eq. (19), using the same angle θ .

Prob. 10. — A telegraph line 300 km. long has a resistance of 6 ohm. per km. There are 15 poles per km. When the receiving circuit is open, and the sending end is connected to a 100-volt battery, the battery ammeter reads 0.13 amp. due to the leakage over the insulators. Compute the average insulation resistance per pole.

Prob. 11. — Extend the foregoing general theory to the case when the insulation conductance g varies along the line as a given continuous function of s , or when it has say two or three different constant values over certain portions of the line. Such a case arises when a different construction is adopted for different parts of the line, or when it rains over a portion of the line.

Prob. 12. — As an application of Prob. 11, investigate the following practical case: A telegraph line extends between cities A and B , and identical batteries and relays are used at both ends. When it rains between A and a certain place C on the line, so that the leakage is increased on the portion AC , determine at which station, A or B , signals sent from the other end of the line are received more clearly.

Note. — The theory given in this chapter may be extended to the case of long alternating-current transmission lines with distributed capacity and inductance. Similar formulæ apply here, except that hyperbolic functions of a complex variable are used. See the author's "Electric Circuit," page 208, and his article entitled "Trigonometric Expressions for the Phenomena Occurring in Long-distance Transmission Lines," *Electrical World*, Vol. 66 (1915), p. 857; literature references are given in both places.

CHAPTER V.

AVERAGE AND EFFECTIVE VALUES OF VOLT- AGES AND CURRENTS.

IN a great many practical cases electric currents, voltages, and magnetic fluxes vary periodically with time, as shown in Fig. 14.

The quantity under consideration, say the voltage across a line, increases, then decreases, is reversed,

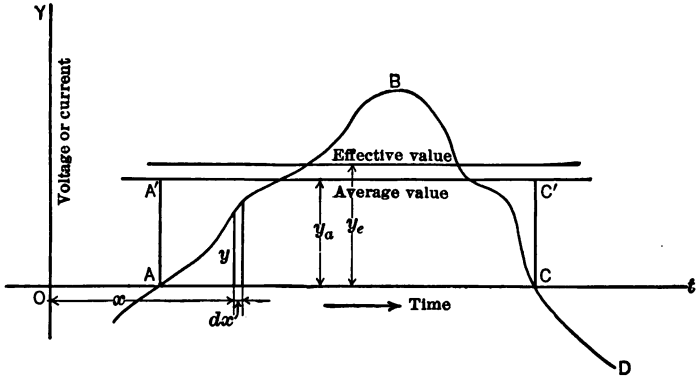


FIG. 14. — An alternating voltage.

increases in the opposite direction, and decreases to zero again, as is shown in part by the curve $ABCD$. In some problems only the average value of the alternating quantity is required, and not the variable instantaneous values. The average value $AA' = y_a$ is defined as such a value that the area between the curve ABC and the axis of abscissæ is equal to the area of the rectangle $AA'C'C$. In other words,

$$AC \times y_a = \int_{OA}^{OC} y \, dx. \quad . \quad . \quad . \quad (1)$$

Prob. 1. — The electromotive force of an alternator varies approximately according to a straight-line law (Fig. 15), the maximum value b being 3000 volts; what is the average value?

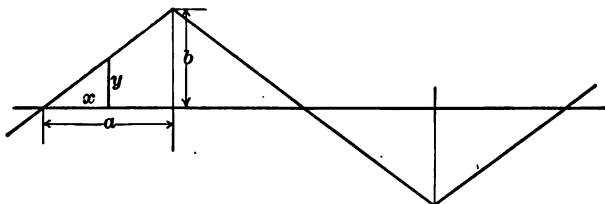


FIG. 15. — An electromotive force which varies periodically according to a straight-line law.

Prob. 2. — A current varies with time as is shown in Fig. 16; it is equal to zero for p per cent of an alternation, and has a constant positive or negative value for the rest of the alternation. Let

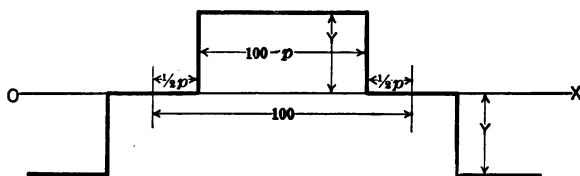


FIG. 16. — A rectangular alternating wave.

the average value be equal to q per cent of the maximum value. Derive an equation, and plot a curve between p and q .

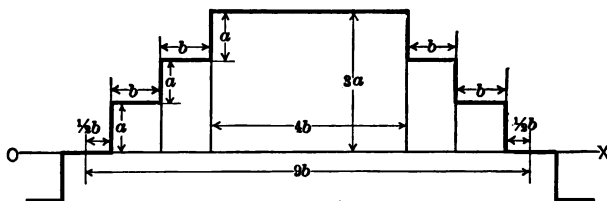


FIG. 17. — A stepped curve of current or voltage.

Prob. 3. — The electromotive force of an alternator varies according to a stepped curve shown in Fig. 17. Find the average value.

Prob. 4. — A current varies with time according to a sine curve (Fig. 18), the equation of which is $i = I \sin 2\pi ft$, where I is the instantaneous maximum value, or the amplitude of the wave, and f is the frequency, or the number of complete cycles of the wave

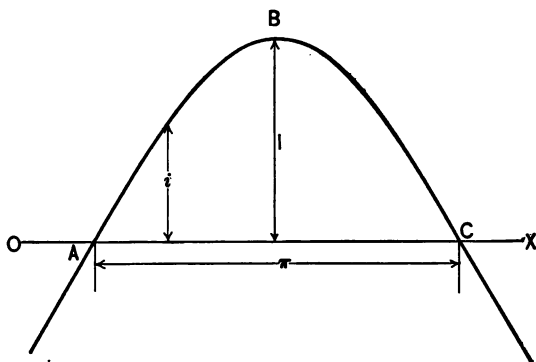


FIG. 18. — A sinusoidal current.

per second. Find the average value of the current. *Solution.* — For a complete cycle the angle $x = 2\pi ft$ increases by 2π ; hence, in terms of x , the base AC of a half-wave is equal to π . Therefore, according to Eq. (1), we have

$$\pi \cdot i_a = \int_0^\pi i \, dx = I \int_0^\pi \sin x \, dx = 2I,$$

from which $i_a = (2/\pi) I = 0.637 I$.

Prob. 5. — A non-sinusoidal curve, such as is shown in Figs. 14 to 17, may be represented by an infinite series of sinusoidal expressions (Fourier's series) of certain frequencies or base lengths.* In a special case of a symmetrical curve this series becomes

$$y = A_1 \sin x + A_3 \sin 3x + A_5 \sin 5x + \text{etc.}, \quad (2)$$

where $x = 2\pi ft$, as in the preceding problem. Show that the average value of the ordinate

$$y_a = (2/\pi) [A_1 + \frac{1}{3} A_3 + \frac{1}{5} A_5 + \dots]. \quad (3)$$

Prob. 6. — What is the average value of a more general curve $y = A_1 \sin(x + \alpha_1) + A_3 \sin(3x + \alpha_3) + A_5 \sin(5x + \alpha_5) + \text{etc.}$, (4) where $\alpha_1, \alpha_3, \alpha_5$, etc., are given angles?

* For further details see the author's "Experimental Electrical Engineering," Vol. II, p. 222, and his "Electric Circuit," p. 42.

Prob. 7. — Find the average ordinate of the parabola (Fig. 19), given its width $2a$ and its height b . *Solution.* — Let the origin be at O . According to a fundamental property of the parabola, we have for a point C , $BC = p\overline{AB}^2$, where p is a constant. The origin

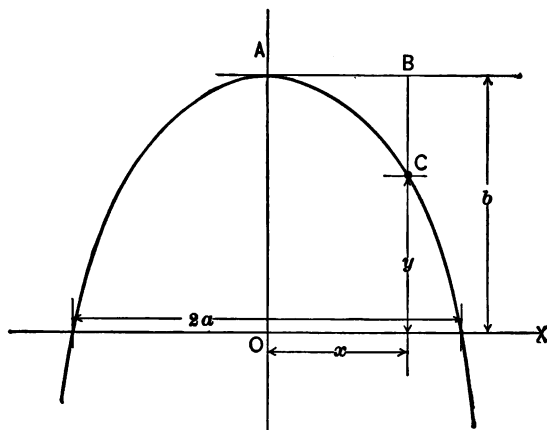


FIG. 19. — A parabolic curve.

being at O , the above equation can be written in the form $(b - y) = px^2$. The constant p is determined from the condition that $x = a$ when $y = 0$; hence, $b = pa^2$, and the equation of the parabola is

$$y = b \left[1 - \frac{x^2}{a^2} \right].$$

The average ordinate is determined from the expression

$$a \cdot y_a = b \int_0^a \left(1 - \frac{x^2}{a^2} \right) dx = \frac{2}{3} ab,$$

or $y_a = \frac{2}{3} b$. This result could have been written directly, knowing that the area enclosed by a parabola is equal to two thirds of the area of the circumscribed rectangle.

Prob. 8. — Solve the preceding problem for the case when the given curve is an hyperbola.

Prob. 9. — Solve Prob. 7 when the given curve is a parabola of the n th degree.

Prob. 10. — Solve Prob. 7 when the given curve is part of an ellipse.

Prob. 11. — Solve Prob. 7 when the given curve has a point of inflection as shown in curve No. 1 of Fig. 20. Consult some work on analytic geometry for curves of such a shape.

Prob. 12. — Arrange curves similar in general shape to those shown in No. 2 and No. 3 (Fig. 20), using portions of well-known curves, such as a parabola, an hyperbola, an ellipse, a sine-curve, etc., and determine their average values.

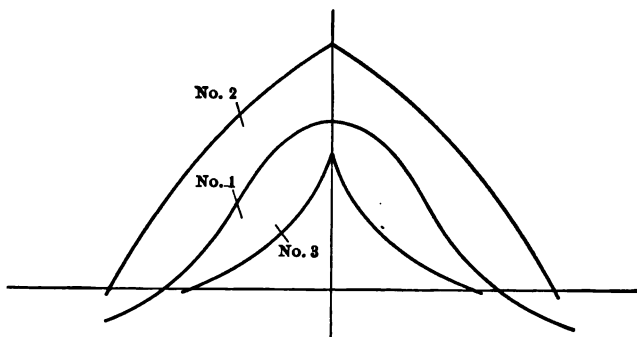


FIG. 20. — Examples of wave forms.

Beside the average value of an alternating current or voltage, another value is widely used in practice, known as the quadratic average or the mean effective value. In order to see its significance, let a variable current i flow through a conductor of resistance r . The instantaneous energy dW converted into heat during an infinitesimal interval dt of time is $dW = i^2 r dt$, and the heat energy developed during an interval of time T is

$$W = \int_0^T i^2 r dt.$$

The value of a steady current, which, flowing through the same conductor for the same length of time T would develop the same amount of heat energy W as

the variable current i , is called the effective value (i_e) of the variable current i . Hence, by definition

$$ri_e^2 T = \int_0^T i^2 r dt,$$

so that

$$i_e^2 = \frac{1}{T} \int_0^T i^2 dt. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This is sometimes written in the form

$$i_{eff} = \sqrt{\frac{1}{T} \int_0^T i^2 dt}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

although the simpler form (5) is usually preferable.

Prob. 13. — Calculate the effective value of the electromotive force shown in Fig. 15. *Solution.* —

$$ay_e^2 = \int_0^a y^2 dx.$$

From similar triangles, $y/b = x/a$, so that

$$ay_e^2 = \int_0^a (bx/a)^2 dx = \frac{1}{3} ab^2,$$

from which $y_e = b/\sqrt{3} = 1732$ volts.

Prob. 14. — Show that the result derived in the preceding problem could be written directly, knowing that the area between a parabola and the tangent at the vertex is equal to one third of the area of the corresponding rectangle.

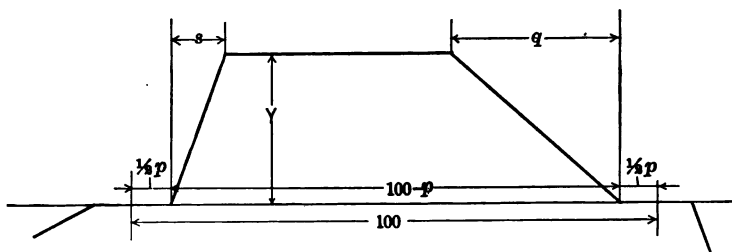


FIG. 21. — A trapezoidal alternating wave.

Prob. 15. — Solve Prob. 2 for the effective value.

Prob. 16. — Solve Prob. 3 for the effective value.

Prob. 17. — Find the effective value of the trapezoidal wave shown in Fig. 21, and investigate the influence of the relative values

of percentages p , q , and s . Also consider the cases when some or all of these percentages are equal to zero.

Prob. 18. — Find the effective value of the sine wave shown in Fig. 18. *Solution.* —

$$\pi \cdot i_e^2 = I^2 \int_0^\pi \sin^2 x \, dx = \frac{1}{2} \pi I^2,$$

from which $i_e = I/\sqrt{2}$.

Prob. 19. — Prove that the effective value of the curve given in Prob. 5 is determined from the equation

$$2y_e^2 = A_1^2 + A_3^2 + A_5^2 + \text{etc.} \quad (7)$$

Hint. — The expression for y^2 consists of two kinds of terms, viz., the squares

$$\sin^2 x, \sin^2 3x, \sin^2 5x, \text{etc.},$$

and the products

$$\sin x \sin 3x, \sin x \sin 5x, \sin 3x \sin 5x, \text{etc.}$$

The terms containing squares of sines are integrated as in the preceding problem. The terms containing products of sines vanish in integration, as is proved in the following problem.

Prob. 20. — As an auxiliary theorem to the preceding problem prove that

$$\int_0^\pi \sin mx \cdot \sin nx \cdot dx = 0,$$

when m and n are odd integers. *Solution.* — The familiar relation, $2 \sin mx \sin nx = \cos (m - n)x - \cos (m + n)x$, gives

$$\begin{aligned} \int_0^\pi \sin mx \sin nx \, dx &= \frac{1}{2} \int_0^\pi \cos (m - n)x \, dx - \frac{1}{2} \int_0^\pi \cos (m + n)x \, dx \\ &= \frac{1}{2} \left[\frac{\sin (m - n)x}{(m - n)} \right]_0^\pi - \frac{1}{2} \left[\frac{\sin (m + n)x}{(m + n)} \right]_0^\pi = 0 - 0 = 0. \end{aligned}$$

Prob. 21. — Show that formula (7) holds true for the curve expressed by Eq. (4).

Prob. 22. — An irregular periodic curve is found to be approximately represented by the equation

$$y = 20 \sin x + 6 \sin (3x + 30^\circ) + 4 \sin (5x - 45^\circ).$$

Compute the mean quadratic ordinate by three methods: (a) Analytically; (b) graphically in rectangular coördinates; (c) graphically in polar coördinates. Show that the results are the same.*

* For the graphical solutions see the author's "Electric Circuit," pp. 48 to 54.

Prob. 23. — Find the effective value of the parabolic curve given in Prob. 7.

Prob. 24. — Find the effective value of the hyperbola the equation of which is supposed to be known from Prob. 8.

Prob. 25. — Find the effective values of the curves mentioned in Probs. 9 to 12.

Prob. 26. — An electric circuit is connected to a source of partially rectified current, that is, a current consisting of a direct-current component i_0 and of an alternating current $I \sin x$, so that the total current is $y = i_0 + I \sin x$. Find the average and the effective values of the current.

Prob. 27. — Referring to the preceding problem, two ammeters are connected in the circuit: a hot-wire instrument which reads effective values, and a moving-coil permanent-magnet instrument which indicates average values. What relationship must exist between the values of i_0 and I in order that the indications of both ammeters be the same?

CHAPTER VI.

EFFICIENCY AND LOSSES IN ELECTRICAL MACHINERY.

THE problems collected in this chapter are maxima and minima problems pertaining to the best dimensions of, and the best connections in, electric batteries, generators, motors, and transformers. The mathematics in these problems is rather simple, and many of them can be solved even without differentiation, by applying one of the theorems in the Appendix. These problems are put at the end of the part on "Electrical Engineering," because they presuppose more technical knowledge of electricity than the problems in the other chapters, and for this reason are perhaps of less general interest.

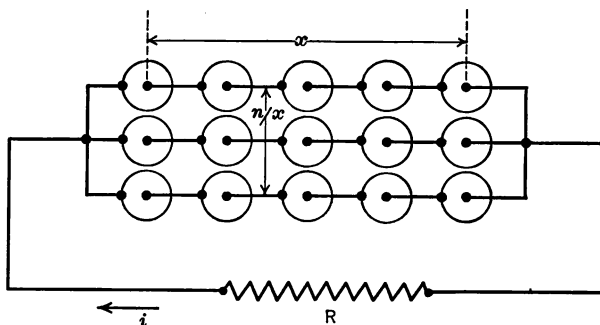


FIG. 22. — A series-parallel connection of electric cells.

Prob. 1. — It is required to arrange n cells of an electric battery in such a series-parallel combination (Fig. 22) as to obtain a maximum current through a given external resistance R . The electromotive force of each cell is e , its internal resistance is r . *Solution.* —

Let x cells be connected in series and n/x cells in parallel. The total electromotive force of the battery is xe ; the total internal resistance is $\frac{rx}{n/x} = \frac{rx^2}{n}$. Hence, the current

$$i = \frac{xe}{R + \frac{rx^2}{n}} \quad \dots \dots \dots (1)$$

Dividing the numerator and the denominator of this fraction by x gives

$$i = \frac{e}{\frac{R}{x} + \frac{rx}{n}} \quad \dots \dots \dots (2)$$

This expression is a maximum when the denominator is a minimum. Theorem 1 in the Appendix applies to the denominator, because

$$\frac{R}{x} \cdot \frac{rx}{n} = \frac{Rr}{n} = \text{Const.},$$

so that it is a minimum when $R/x = rx/n$, or

$$R = \frac{rx^2}{n} \quad \dots \dots \dots (3)$$

In other words, the current is a maximum when the external resistance is equal to the internal resistance of the battery.

Prob. 2. — Derive result (3) by differentiating the denominator of Eq. (2) directly, and show that it corresponds to a minimum and not to a maximum.

Prob. 3. — Deduce solution (3) by differentiating directly Eq. (1); investigate the sign of the second derivative.

Prob. 4. — Prove that the efficiency of a battery, when it delivers a maximum current through a given external resistance, is only 50 per cent. *Note.* — Efficiency is the ratio of the power Ei delivered to the external circuit, to the power E_0i developed in the battery. E is the terminal voltage of the battery, and E_0 is its electromotive force on open circuit.

Prob. 5. — A telegraph station has a battery of 364 Daniell cells, the internal resistance of each cell is 1 ohm, and the electromotive force of each cell is 1.08 volt. What is the best way to connect these cells, so as to obtain a maximum current through a circuit of about 40 ohms resistance, and what will be the value of the current in the line and through each cell?

Prob. 6. — Assume the cells given in the preceding problem to be connected, first all in series, then in two parallel branches, in three branches, four branches, etc. Plot curves of line current and efficiency to number of branches as abscissæ, and thus prove that in order to obtain a maximum current the cells must be connected in three parallel branches.

Prob. 7. — Let it be required to solve Prob. 5 when 420 cells are available. According to Eq. (3), we find that 130 cells must be connected in series. Connecting three rows in parallel, the number of superfluous or idle cells is $420 - 3 \times 130 = 30$. It stands to reason, however, that a larger current could be obtained by using these 30 cells, in sets of 10, in series with each row, so that the true practical solution seems to be $140 \times 3 = 420$ cells. Explain this discrepancy and give a definite rule for using Eq. (3) in such cases.

Prob. 8. — An adjustable electric heater is placed at the end of a long line of resistance r ; the generator voltage is e . At what value of current does the heater receive a maximum of electric power? *Solution.* — Let the current be i ; then the voltage drop in the line is ir and the voltage across the heater is $e - ir$. Thus, the power delivered to the heater is $i(e - ir)$ and this expression must be a maximum. Multiplying this expression by a constant quantity r we see that

$$ri + (e - ri) = \text{Const.}$$

hence Theorem 1 applies to it, and the power is a maximum when $ri = e - ri$, or

$$ri = \frac{1}{2}e. \quad (4)$$

In other words, the delivered power is a maximum when one half of the applied voltage is consumed in the line.

Prob. 9. — Referring to the preceding problem, let the generator voltage be 100 volts and the resistance of the line 2.5 ohms. Plot values of power delivered to the heater and efficiency of the transmission against current as abscissæ. The resistance of the heater is adjustable between zero and 10 ohms. From these curves show that for maximum power, condition (4) must be satisfied.

Prob. 10. — Define mathematically the curves obtained in the preceding problem, in terms of analytic geometry, and reduce their equations to the simplest possible form.

Prob. 11. — At what value of the counter-electromotive force is the power output of a series-wound direct-current motor a maxi-

mum, and what is its efficiency in this case, neglecting all the losses except the copper loss? *Solution.* — Let E be the voltage applied at the terminals of the motor and e its counter-e.m.f. The current through the armature is $i = (E - e)/R$, where R is the total resistance of the motor. The power output converted into mechanical work is $ei = e(E - e)/R$, because Ei is the input and $i(E - e)$ is the copper loss in the armature itself. Since $e + (E - e) = \text{Const.}$, the condition for a maximum is $e = E - e$, or $e = \frac{1}{2}E$. The efficiency is 50 per cent. $\dagger$

Prob. 12. — Solve Prob. 11, assuming that in addition to the copper loss there is a constant loss of K watts, to account for the iron loss and friction.

Prob. 13. — A shunt-wound generator (Fig. 23) is operating at a constant terminal voltage e . The resistance of the armature is r , that of the shunt winding is R . At what current i is the efficiency of the machine a maximum? Neglect the friction and iron loss,

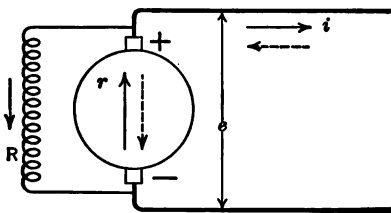


FIG. 23. — A shunt-wound generator or motor.

and assume the resistance of the field circuit to be practically constant within the operating limits. *Solution.* — The field current is e/R , so that the current through the armature is $i + e/R$. The loss of power in the field winding is e^2/R , that in the armature is $(i + e/R)r$. Hence, the efficiency of the machine is

$$\eta = \frac{ei}{ei + e^2/R + (i + e/R)r}.$$

Instead of taking a derivative of this expression, we first divide its numerator and denominator by i , so as to make the numerator a constant. We thus obtain

$$\eta = \frac{e}{e + e^2/(Ri) + (i + e/R)^2(r/i)}.$$

This expression becomes a maximum when the denominator is a minimum. Omitting the constant terms, the denominator may be written in the form

$$(e^2/Ri) (1 + r/R) + ri = \min.$$

The product of these two terms is constant, therefore they must be equal to each other. This condition gives an equation which can be solved for i .

Prob. 14. — Referring to the preceding problem, let K be a constant iron and friction loss in watts. Show that the efficiency is a maximum when the external current is

$$i = \frac{e}{R} \sqrt{\frac{R}{r} \left(1 + \frac{KR}{e^2} \right)} + 1. \quad \dots \quad (5)$$

Prob. 15. — Plot a curve of per cent efficiency against line current for a twenty-kilowatt, shunt-wound generator operating at a terminal voltage of 550 volts. The armature resistance is 1.7 ohms, the field resistance is 600 ohms, and the no-load losses K amount to 800 watts. Show that the curve is quite flat and that the maximum efficiency is at a current corresponding to expression (5).

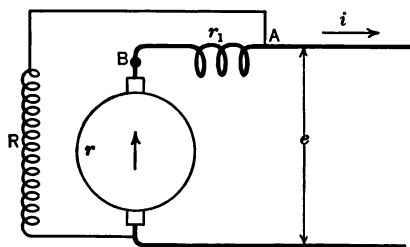


FIG. 24. — A compound-wound generator.

Prob. 16. — Modify the solution of Probs. 13 and 14 for a compound-wound generator (Fig. 24), the resistance of the series winding being r_1 .

Prob. 17. — Modify the solution of Prob. 16 for a machine with a short shunt. This means that the shunt-field winding R (Fig. 24) is connected to the main circuit at B , instead of at A .

Prob. 18. — A shunt-wound motor (Fig. 23) has an armature resistance r , a field resistance R , and a no-load loss of K watts. What is the maximum theoretical output P of the motor at a line

voltage of e volts (apart from the limits imposed by sparking and overheating)? *Solution.* — Let i be the armature current; the output is equal to the input minus the losses, so that

$$\text{Output } P = (i + e/R) e - (e^2/R - i^2 r - K),$$

or

$$P = ie - i^2 r - K. \quad (6)$$

This expression becomes a maximum when $i = \frac{1}{2} e/r$ (see Prob. 8).

Prob. 19. — At what value of i does the motor in the foregoing problem operate at a maximum efficiency? *Hint.* — Use the total current $I = i + e/R$ as the independent variable, and reduce the expression for efficiency to a sum of two terms the product of which is constant.

Prob. 20. — Find the coördinates of the vertex of the parabola expressed by Eq. (6), and its parameter. Write the equation of this parabola in its simplest form taking the vertex for the origin.

Prob. 21. — Prove that a transformer has a maximum efficiency at a load ei at which the copper loss $i^2 r$ is equal to the iron loss K .

$$\begin{aligned} \text{Solution. — Efficiency} &= \frac{\text{output}}{\text{output} + \text{copper loss} + \text{iron loss}} \\ &= \frac{ei}{ei + i^2 r + K} = \frac{e}{e + ir + K/i}. \end{aligned}$$

The denominator must be a minimum, and since $ir (K/i) = \text{Const.}$, $ir = K/i$, or $i^2 r = K$.

Prob. 22. — Is the preceding result modified if the losses are represented by the formula $i^2 r + ai + K$, where a is a constant?

Note. — The power loss caused by the voltage drop across the brushes of a generator or a motor may be represented by the term ai .

Prob. 23. — The efficiency of a transformer must be a maximum at three quarters of the rated load and must be equal to 95 per cent. What will its efficiency be at full load?

Prob. 24. — The dimensions of the iron core of a transformer are given (Fig. 25), and also the available space for the copper, that is, the inside diameter D_1 of the winding and its outside diameter D_2 . Determine the thickness x of the inside coil (say the primary winding) and the thickness y of the outside secondary coil such as would make the total copper loss a minimum. *Solution.* — Let n_1 and n_2 be the numbers of turns of the primary and of the secondary windings, i_1 and i_2 the corresponding currents, and q_1 and q_2 the cross sections of the wire. The resistance of the primary

winding is $\rho n_1 \pi (D_1 + x)/q_1$, where ρ is the resistivity of the material and $\pi (D_1 + x)$ is the mean length of one turn. Therefore the condition for minimum copper loss is

$$\rho n_1 \pi (D_1 + x) i_1^2 / q_1 + \rho n_2 \pi (D_2 - y) i_2^2 / q_2 = \min. \quad (7)$$

The cross section of the primary wire is $q_1 = \alpha x h / n_1$, where h is the height of the coil and α is the "space factor" of the coil. It means

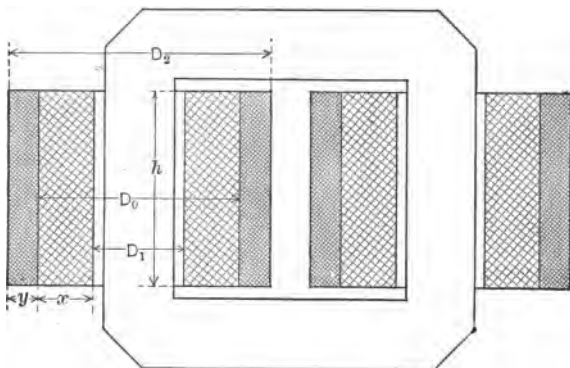


FIG. 25. — A core-type transformer with concentric coils.

that while the gross cross section of the coil is xh , only a part α is utilized for the copper, the rest being taken by the insulation and by the air spaces. The useful cross section $\alpha x h$ comprises n_1 wires, hence the cross section of each wire is $\alpha x h / n_1$. Similarly, $q_2 = \alpha y h / n_2$. Substituting the values of q_1 and q_2 in Eq. (7) and omitting all of the constant factors, we get

$$n_1^2 i_1^2 (D_1 + x) / x + n_2^2 i_2^2 (D_2 - y) / y = \min.$$

It is proved in the theory of the transformer that $n_1 i_1 = n_2 i_2$, with an accuracy sufficient for most practical purposes. Hence, the foregoing equation is simplified to

$$D_1 / x + D_2 / y = \min. \quad (8)$$

This expression, together with the condition

$$x + y = \frac{1}{2} (D_2 - D_1) = \text{Const.}, \quad (9)$$

is sufficient for the solution of the problem. Differentiating these equations we get

$$\begin{aligned} D_1 x^{-2} dx + D_2 y^{-2} dy &= 0; \\ dx + dy &= 0. \end{aligned}$$

Substituting the value $dy = -dx$ in the first of these equations, and rearranging the terms, we obtain

$$x : y = \sqrt{D_1} : \sqrt{D_2}. \quad \dots \quad (10)$$

It is left to the student to prove that the values of x and y so obtained correspond to a minimum of the function, and not to any other stationary value; see Note at the end of Chapter II.

Prob. 25. — Using the results of the preceding problem, show that the diameter D_0 of the cylindrical surface of separation (Fig. 25) between the primary and the secondary coils is the geometrical mean of the diameters D_1 and D_2 .

Prob. 26. — Show that when the condition of minimum copper loss in Prob. 24 is satisfied, equal amounts of Joulean heat, i^2r , are developed in both windings.

Prob. 27. — In a transformer $D_1 = 36$ cm., $D_2 = 68$ cm., $h = 45$ cm. The high-tension coil is on the outside; it has 2000 turns and carries a current of 10 amps. The low-tension current is 200 amps. Calculate the cross sections of the primary and the secondary wire, assuming the space factor to be 50 per cent for both coils.

Prob. 28. — Solve Prob. 24 for the case when the space factors, α_1 and α_2 , of the two windings are different.

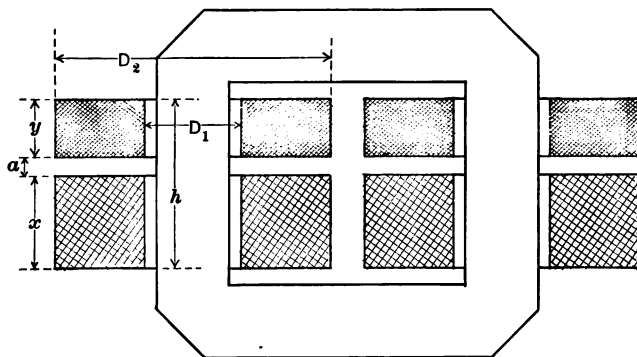


FIG. 26. — A core-type transformer with flat coils.

Prob. 29. — In practice, a cylindrical layer of insulation is interposed between the two coils. Let the thickness of this insulation be a cm. How will this new condition modify the solution of Prob. 24?

Prob. 30. — Solve Prob. 24 when the two coils are placed side by side (Fig. 26) instead of being concentric. The unknown quan-

tities are the heights, x and y , of the coils. A layer of insulation, a , is to be taken into account between the coils.

Prob. 31. — The magnetic circuit of an alternating-current apparatus (Fig. 27) has a cross section Q_1 on the part l_1 of its length, and a cross section Q_2 on the rest, l_2 , of the length. Select Q_1 and Q_2

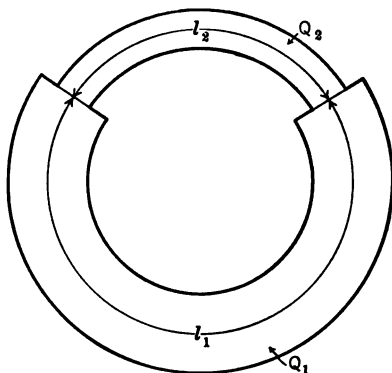


FIG. 27. — An iron core with two different cross sections.

so that the hysteresis loss will be a minimum for a given total weight of the core. *Solution.* — The hysteresis loss per cubic cm. of the core may be assumed to be proportional to some n th power of the flux density B ; therefore

$$Q_1 l_1 B_1^n + Q_2 l_2 B_2^n = \min.; \quad \dots \quad (11)$$

the limiting condition being that the volume of iron

$$V = l_1 Q_1 + l_2 Q_2 = \text{Const.} \quad \dots \quad (12)$$

Since the same flux Φ is produced in all parts of the core, we have

$$\Phi = B_1 Q_1 = B_2 Q_2 = \text{Const.} \quad \dots \quad (13)$$

Substituting the values of B_1 and B_2 from Eq. (13) into Eq. (11), and omitting the constant factors, we get

$$\frac{l_1}{Q_1^{n-1}} + \frac{l_2}{Q_2^{n-1}} = \min. \quad \dots \quad (14)$$

Using the method of indeterminate coefficients, we have, from Eqs. (14) and (12),

$$\frac{l_1}{Q_1^{n-1}} + \frac{l_2}{Q_2^{n-1}} + \lambda (l_1 Q_1 + l_2 Q_2) = \min.,$$

or, taking partial derivatives, $-(n-1) l_1/Q_1^n + \lambda_1 = 0$, and $-(n-1) l_2/Q_2^n + \lambda_2 = 0$. From these expressions we readily find that $Q_1 = Q_2$.

Prob. 32. — Solve the preceding problem by direct substitution, eliminating one of the Q 's, so as to avoid the use of partial derivatives.

Prob. 33. — Solve Prob. 31 with the requirement that the total iron loss consisting of hysteresis and of eddy currents be a minimum. Assume that the loss per cubic cm. of the core can be represented by the formula $\alpha B^m + \beta B^n$. *Hint.* $Q_1 l_1 (\alpha B_1^m + \beta B_1^n) + Q_2 l_2 (\alpha B_2^m + \beta B_2^n) = \min$.

Prob. 34. — A transformer core is given, and the space available for the windings is known. It is required to determine the flux density B in the core so as to have a minimum amount of loss for a given output. *Solution.* — From the theory of the transformer it is known that the copper loss at a given current and with a given volume of winding is inversely proportional to the square of the flux density; or

$$\text{Copper loss} = CB^{-2}, \dots \dots \dots (15)$$

where C is a constant. The iron loss may be assumed to be proportional to a certain power m of the magnetic induction, or,

$$\text{Iron loss} = \alpha B^m. \dots \dots \dots (16)$$

Thus

$$CB^{-2} + \alpha B^m = \min., \dots \dots \dots (17)$$

which is easily solved for B . The following is an interesting interpretation of the result: Differentiating Eq. (17) and multiplying the result by B we obtain $-2CB^{-2} + m\alpha B^{m-1} = 0$. In other words, B must be selected so that

$$\text{Copper loss} = \frac{1}{2} m [\text{iron loss}]. \dots \dots \dots (18)$$

Prob. 35. — Referring to the foregoing problem let the iron loss be represented by the usual formula, $\alpha B^{1.6} + \beta B^2$, where the first term represents the hysteresis loss and the second term the eddy-current loss. Show that the total loss for a given output is a minimum, when the condition is fulfilled that

Copper loss = 80 per cent of hysteresis loss + full eddy current loss.
(Arnold's rule.)

Prob. 36. — A part of the magnetic circuit of an electrical apparatus (Fig. 28) is to be made of steel laminations, and must be of a given thickness h , and width a . The thickness of the insulation between the punchings is δ . Determine the thickness x of the laminations such that the iron loss for a given alternating flux be a minimum. *Solution.*

— By taking a few thick laminations the space lost for insulation is reduced, hence the flux density and the hysteresis loss are reduced. On the other hand the eddy-current loss is increased, because

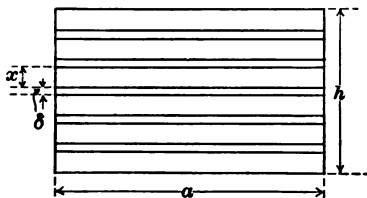


FIG. 28. — A laminated magnetic circuit.

this loss per unit volume is proportional to the square of the thickness of laminations. Thus, there must be a thickness x for which the total loss is a minimum. Let B_0 equal the flux density and V_0 the volume of iron if a piece of the same dimensions h and a were made out of solid iron. Let y be the space factor of the laminated core, that is, the part actually occupied by iron. For instance, if the insulation takes 10 per cent of h , $y = 0.90$. In terms of x and δ we have

$$y = x/(x + \delta). \quad (19)$$

The actual flux density, B , in the laminated core is greater than B_0 , namely, $B = B_0/y$. The actual volume of iron in the laminated core is $V = V_0 y$, and is less than V_0 . According to Prob. 33, the condition for a minimum iron loss is

$$V (\alpha B^m + \beta B^n) = \min.$$

Substituting the values of V and B in terms of y , we get

$$P y^{1-m} + Q y^{1-n} = \min., \quad (20)$$

where P and Q are known constants.

Prob. 37. — In the preceding problem, let $y = 0.85$, $m = 1.6$, $n = 2$, $\delta = 0.05$ mm., and $B_0 = 15,000$ lines per sq. cm. What is the ratio of α to β when the core satisfies the condition of minimum iron loss?

APPENDIX.

IN many practical problems the product of two variables must be constant, while their sum must be a minimum. The mathematical problem is therefore to find the values of u and v that satisfy the conditions

$$\begin{aligned} uv &= C = \text{const.}; \\ u + v &= \text{min.} \end{aligned}$$

Substituting the value of v from the first equation into the second, we obtain

$$u + C/u = \text{min.}$$

Equating the first derivative to zero, we get

$$1 - C/u^2 = 0,$$

or $u = \sqrt{C}$; consequently $v = \sqrt{C}$. We thus arrive at the following result:

Theorem 1. — When the product of two variables is constant, their sum is a minimum when the variables are equal to each other.

Knowing this theorem, it is not necessary to differentiate in each particular problem, but the solution may be written directly. See, for example, Prob. 1 in Chapter III.

Sometimes the variables satisfy a more general condition

$$u^m v^n = C = \text{const.}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the expression to be made a minimum is of the form

$$u^p + v^q = \text{min.}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where m , n , p and q are known constants.

The total differential of the first expression is equal to zero, because the expression itself is a constant; the total differential of the second expression is also equal to zero, because the expression must be a minimum. We thus have

$$\begin{aligned} mu^{m-1}v^n du + nu^mv^{n-1} dv &= 0; \\ pu^{p-1} du + qv^{q-1} dv &= 0. \end{aligned}$$

Eliminating du and dv , and simplifying, we get

$$\frac{u^p}{v^q} = \frac{mq}{np}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The values of u and v are obtained by solving Eqs. (1) and (3) as simultaneous equations. This result we shall call

Theorem 2. — When two variables satisfy condition (1), and expression (2) must be a minimum, the values of the variables must satisfy Eq. (3). For example, Eq. (28) in Chap. III can be solved using this theorem.

In some problems the sum of two variables is constant, and their product must be a maximum; for instance, when it is required to find a rectangle of maximum area, the perimeter being given. The conditions are:

$$\begin{aligned} u + v &= \text{const.}; \\ uv &= \text{max.} \end{aligned}$$

Proceeding as in the proof of Theorem 1, we get

Theorem 3. — When the sum of two variables is constant, their product is a maximum when the variables are equal to each other.

In a more general case we may have

$$u^m + v^n = \text{const.}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$u^p v^q = \text{max.} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Proceeding as in the case of Theorem 2, we derive the following theorem:

Theorem 4. — When two variables satisfy condition (4), and expression (5) must be a maximum, the values of the variables must be such that

$$\frac{u^m}{v^n} = \frac{np}{mq} (6)$$

The student will find throughout the book a number of other problems on maxima and minima in which the actual differentiation can be avoided by using one of these four theorems.

Strictly speaking, in order to complete the proofs it is necessary to show that the critical values of the variables found above correspond to a minimum or a maximum respectively. This may be done by considering the sign of the second derivative, by interpreting the results geometrically, or perhaps by some special reasoning.

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